

# Mathematics

*Academic Mathematics 3201*

**Interim Edition**



*Curriculum Guide*  
**2013**



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## INTRODUCTION

### Background

*The curriculum guide communicates high expectations for students.*

The Mathematics curriculum guides for Newfoundland and Labrador have been derived from *The Common Curriculum Framework for 10-12 Mathematics: Western and Northern Canadian Protocol*, January 2008. These guides incorporate the conceptual framework for Grades 10 to 12 Mathematics and the general outcomes, specific outcomes and achievement indicators established in the common curriculum framework. They also include suggestions for teaching and learning, suggested assessment strategies, and an identification of the associated resource match between the curriculum and authorized, as well as recommended, resource materials.

### Beliefs About Students and Mathematics

*Mathematical understanding is fostered when students build on their own experiences and prior knowledge.*

Students are curious, active learners with individual interests, abilities and needs. They come to classrooms with varying knowledge, life experiences and backgrounds. A key component in developing mathematical literacy is making connections to these backgrounds and experiences.

Students learn by attaching meaning to what they do, and they need to construct their own meaning of mathematics. This meaning is best developed when learners encounter mathematical experiences that proceed from the simple to the complex and from the concrete to the abstract. Through the use of manipulatives and a variety of pedagogical approaches, teachers can address the diverse learning styles, cultural backgrounds and developmental stages of students, and enhance within them the formation of sound, transferable mathematical understandings. Students at all levels benefit from working with a variety of materials, tools and contexts when constructing meaning about new mathematical ideas. Meaningful student discussions provide essential links among concrete, pictorial and symbolic representations of mathematical concepts.

The learning environment should value and respect the diversity of students' experiences and ways of thinking, so that students feel comfortable taking intellectual risks, asking questions and posing conjectures. Students need to explore problem-solving situations in order to develop personal strategies and become mathematically literate. They must come to understand that it is acceptable to solve problems in a variety of ways and that a variety of solutions may be acceptable.

# Program Design and Components

## Affective Domain

*To experience success, students must learn to set achievable goals and assess themselves as they work toward these goals.*

A positive attitude is an important aspect of the affective domain and has a profound impact on learning. Environments that create a sense of belonging, encourage risk taking and provide opportunities for success help develop and maintain positive attitudes and self-confidence within students. Students with positive attitudes toward learning mathematics are likely to be motivated and prepared to learn, participate willingly in classroom activities, persist in challenging situations and engage in reflective practices.

Teachers, students and parents need to recognize the relationship between the affective and cognitive domains, and attempt to nurture those aspects of the affective domain that contribute to positive attitudes. To experience success, students must learn to set achievable goals and assess themselves as they work toward these goals.

Striving toward success and becoming autonomous and responsible learners are ongoing, reflective processes that involve revisiting, assessing and revising personal goals.

## Goals For Students

*Mathematics education must prepare students to use mathematics confidently to solve problems.*

The main goals of mathematics education are to prepare students to:

- use mathematics confidently to solve problems
- communicate and reason mathematically
- appreciate and value mathematics
- make connections between mathematics and its applications
- commit themselves to lifelong learning
- become mathematically literate adults, using mathematics to contribute to society.

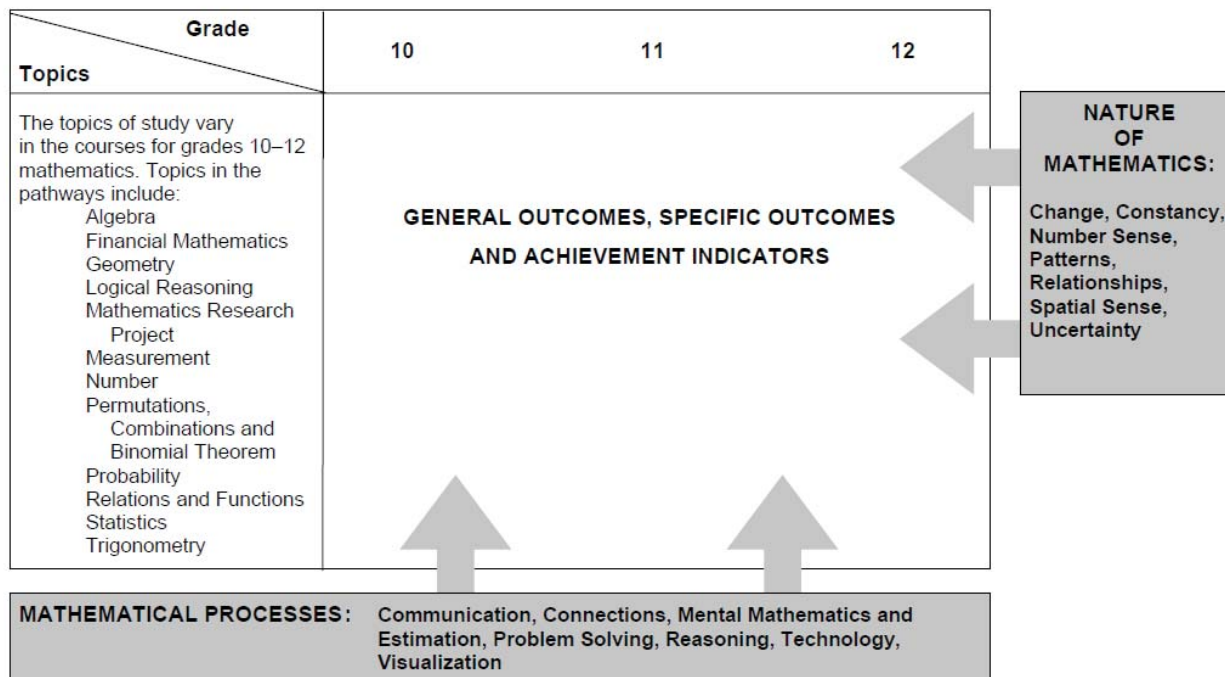
Students who have met these goals will:

- gain understanding and appreciation of the contributions of mathematics as a science, philosophy and art
- exhibit a positive attitude toward mathematics
- engage and persevere in mathematical tasks and projects
- contribute to mathematical discussions
- take risks in performing mathematical tasks
- exhibit curiosity.



## CONCEPTUAL FRAMEWORK FOR 10-12 MATHEMATICS

The chart below provides an overview of how mathematical processes and the nature of mathematics influence learning outcomes.



## Mathematical Processes

- *Communication [C]*
- *Connections [CN]*
- *Mental Mathematics and Estimation [ME]*
- *Problem Solving [PS]*
- *Reasoning [R]*
- *Technology [T]*
- *Visualization [V]*

There are critical components that students must encounter in a mathematics program in order to achieve the goals of mathematics education and embrace lifelong learning in mathematics.

Students are expected to:

- communicate in order to learn and express their understanding
- connect mathematical ideas to other concepts in mathematics, to everyday experiences and to other disciplines
- demonstrate fluency with mental mathematics and estimation
- develop and apply new mathematical knowledge through problem solving
- develop mathematical reasoning
- select and use technologies as tools for learning and for solving problems
- develop visualization skills to assist in processing information, making connections and solving problems.

This curriculum guide incorporates these seven interrelated mathematical processes that are intended to permeate teaching and learning.

## Communication [C]

*Students must be able to communicate mathematical ideas in a variety of ways and contexts.*

Students need opportunities to read about, represent, view, write about, listen to and discuss mathematical ideas. These opportunities allow students to create links between their own language and ideas, and the formal language and symbols of mathematics.

Communication is important in clarifying, reinforcing and modifying ideas, attitudes and beliefs about mathematics. Students should be encouraged to use a variety of forms of communication while learning mathematics. Students also need to communicate their learning using mathematical terminology.

Communication helps students make connections among concrete, pictorial, symbolic, oral, written and mental representations of mathematical ideas.

## Connections [CN]

*Through connections, students begin to view mathematics as useful and relevant.*

Contextualization and making connections to the experiences of learners are powerful processes in developing mathematical understanding. When mathematical ideas are connected to each other or to real-world phenomena, students begin to view mathematics as useful, relevant and integrated.

Learning mathematics within contexts and making connections relevant to learners can validate past experiences and increase student willingness to participate and be actively engaged.

The brain is constantly looking for and making connections. “Because the learner is constantly searching for connections on many levels, educators need to *orchestrate the experiences* from which learners extract understanding ... Brain research establishes and confirms that multiple complex and concrete experiences are essential for meaningful learning and teaching” (Caine and Caine, 1991, p.5).

## Mental Mathematics and Estimation [ME]

*Mental mathematics and estimation are fundamental components of number sense.*

Mental mathematics is a combination of cognitive strategies that enhance flexible thinking and number sense. It is calculating mentally without the use of external memory aids.

Mental mathematics enables students to determine answers without paper and pencil. It improves computational fluency by developing efficiency, accuracy and flexibility.

“Even more important than performing computational procedures or using calculators is the greater facility that students need—more than ever before—with estimation and mental math” (National Council of Teachers of Mathematics, May 2005).

Students proficient with mental mathematics “... become liberated from calculator dependence, build confidence in doing mathematics, become more flexible thinkers and are more able to use multiple approaches to problem solving” (Rubenstein, 2001, p. 442).

Mental mathematics “... provides the cornerstone for all estimation processes, offering a variety of alternative algorithms and nonstandard techniques for finding answers” (Hope, 1988, p. v).

Estimation is used for determining approximate values or quantities or for determining the reasonableness of calculated values. It often uses benchmarks or referents. Students need to know when to estimate, how to estimate and what strategy to use.

Estimation assists individuals in making mathematical judgements and in developing useful, efficient strategies for dealing with situations in daily life.

## Problem Solving [PS]

*Learning through problem solving should be the focus of mathematics at all grade levels.*

Learning through problem solving should be the focus of mathematics at all grade levels. When students encounter new situations and respond to questions of the type, “How would you know?” or “How could you ...?”, the problem-solving approach is being modelled. Students develop their own problem-solving strategies by listening to, discussing and trying different strategies.

A problem-solving activity requires students to determine a way to get from what is known to what is unknown. If students have already been given steps to solve the problem, it is not a problem, but practice. A true problem requires students to use prior learning in new ways and contexts. Problem solving requires and builds depth of conceptual understanding and student engagement.

Problem solving is a powerful teaching tool that fosters multiple, creative and innovative solutions. Creating an environment where students openly seek and engage in a variety of strategies for solving problems empowers students to explore alternatives and develops confident, cognitive mathematical risk takers.

## Reasoning [R]

*Mathematical reasoning helps students think logically and make sense of mathematics.*

Mathematical reasoning helps students think logically and make sense of mathematics. Students need to develop confidence in their abilities to reason and justify their mathematical thinking. High-order questions challenge students to think and develop a sense of wonder about mathematics.

Mathematical experiences in and out of the classroom provide opportunities for students to develop their ability to reason. Students can explore and record results, analyze observations, make and test generalizations from patterns, and reach new conclusions by building upon what is already known or assumed to be true.

Reasoning skills allow students to use a logical process to analyze a problem, reach a conclusion and justify or defend that conclusion.

## Technology [T]

*Technology contributes to the learning of a wide range of mathematical outcomes and enables students to explore and create patterns, examine relationships, test conjectures and solve problems.*

Technology contributes to the learning of a wide range of mathematical outcomes and enables students to explore and create patterns, examine relationships, test conjectures and solve problems.

Technology can be used to:

- explore and demonstrate mathematical relationships and patterns
- organize and display data
- extrapolate and interpolate
- assist with calculation procedures as part of solving problems
- decrease the time spent on computations when other mathematical learning is the focus
- reinforce the learning of basic facts
- develop personal procedures for mathematical operations
- create geometric patterns
- simulate situations
- develop number sense.

Technology contributes to a learning environment in which the growing curiosity of students can lead to rich mathematical discoveries at all grade levels.

## Visualization [V]

*Visualization is fostered through the use of concrete materials, technology and a variety of visual representations.*

Visualization “involves thinking in pictures and images, and the ability to perceive, transform and recreate different aspects of the visual-spatial world” (Armstrong, 1993, p. 10). The use of visualization in the study of mathematics provides students with opportunities to understand mathematical concepts and make connections among them.

Visual images and visual reasoning are important components of number, spatial and measurement sense. Number visualization occurs when students create mental representations of numbers.

Being able to create, interpret and describe a visual representation is part of spatial sense and spatial reasoning. Spatial visualization and reasoning enable students to describe the relationships among and between 3-D objects and 2-D shapes.

Measurement visualization goes beyond the acquisition of specific measurement skills. Measurement sense includes the ability to determine when to measure, when to estimate and which estimation strategies to use (Shaw and Cliatt, 1989).

## Nature of Mathematics

- *Change*
- *Constancy*
- *Number Sense*
- *Patterns*
- *Relationships*
- *Spatial Sense*
- *Uncertainty*

Mathematics is one way of trying to understand, interpret and describe our world. There are a number of components that define the nature of mathematics and these are woven throughout this curriculum guide. The components are change, constancy, number sense, patterns, relationships, spatial sense and uncertainty.

## Change

*Change is an integral part of mathematics and the learning of mathematics.*

It is important for students to understand that mathematics is dynamic and not static. As a result, recognizing change is a key component in understanding and developing mathematics.

Within mathematics, students encounter conditions of change and are required to search for explanations of that change. To make predictions, students need to describe and quantify their observations, look for patterns, and describe those quantities that remain fixed and those that change. For example, the sequence 4, 6, 8, 10, 12, ... can be described as:

- the number of a specific colour of beads in each row of a beaded design
- skip counting by 2s, starting from 4
- an arithmetic sequence, with first term 4 and a common difference of 2
- a linear function with a discrete domain

(Steen, 1990, p. 184).

## Constancy

*Constancy is described by the terms stability, conservation, equilibrium, steady state and symmetry.*

Different aspects of constancy are described by the terms stability, conservation, equilibrium, steady state and symmetry (AAAS-Benchmarks, 1993, p.270). Many important properties in mathematics and science relate to properties that do not change when outside conditions change. Examples of constancy include the following:

- The ratio of the circumference of a teepee to its diameter is the same regardless of the length of the teepee poles.
- The sum of the interior angles of any triangle is  $180^\circ$ .
- The theoretical probability of flipping a coin and getting heads is 0.5.

Some problems in mathematics require students to focus on properties that remain constant. The recognition of constancy enables students to solve problems involving constant rates of change, lines with constant slope, direct variation situations or the angle sums of polygons.

## Number Sense

*An intuition about number is the most important foundation of a numerate child.*

Number sense, which can be thought of as intuition about numbers, is the most important foundation of numeracy (British Columbia Ministry of Education, 2000, p.146).

A true sense of number goes well beyond the skills of simply counting, memorizing facts and the situational rote use of algorithms. Mastery of number facts is expected to be attained by students as they develop their number sense. This mastery allows for facility with more complex computations but should not be attained at the expense of an understanding of number.

Number sense develops when students connect numbers to their own real-life experiences and when students use benchmarks and referents. This results in students who are computationally fluent and flexible with numbers and who have intuition about numbers. The evolving number sense typically comes as a by product of learning rather than through direct instruction. It can be developed by providing rich mathematical tasks that allow students to make connections to their own experiences and their previous learning.

## Patterns

*Mathematics is about recognizing, describing and working with numerical and non-numerical patterns.*

Mathematics is about recognizing, describing and working with numerical and non-numerical patterns. Patterns exist in all strands of mathematics.

Working with patterns enables students to make connections within and beyond mathematics. These skills contribute to students' interaction with, and understanding of, their environment.

Patterns may be represented in concrete, visual or symbolic form. Students should develop fluency in moving from one representation to another.

Students must learn to recognize, extend, create and use mathematical patterns. Patterns allow students to make predictions and justify their reasoning when solving routine and non-routine problems.

Learning to work with patterns in the early grades helps students develop algebraic thinking, which is foundational for working with more abstract mathematics.

## Relationships

*Mathematics is used to describe and explain relationships.*

Mathematics is one way to describe interconnectedness in a holistic worldview. Mathematics is used to describe and explain relationships. As part of the study of mathematics, students look for relationships among numbers, sets, shapes, objects and concepts. The search for possible relationships involves collecting and analyzing data and describing relationships visually, symbolically, orally or in written form.

## Spatial Sense

*Spatial sense offers a way to interpret and reflect on the physical environment.*

Spatial sense involves visualization, mental imagery and spatial reasoning. These skills are central to the understanding of mathematics.

Spatial sense is developed through a variety of experiences and interactions within the environment. The development of spatial sense enables students to solve problems involving 3-D objects and 2-D shapes and to interpret and reflect on the physical environment and its 3-D or 2-D representations.

Some problems involve attaching numerals and appropriate units (measurement) to dimensions of shapes and objects. Spatial sense allows students to make predictions about the results of changing these dimensions; e.g., doubling the length of the side of a square increases the area by a factor of four. Ultimately, spatial sense enables students to communicate about shapes and objects and to create their own representations.



## Uncertainty

*Uncertainty is an inherent part of making predictions.*

In mathematics, interpretations of data and the predictions made from data may lack certainty.

Events and experiments generate statistical data that can be used to make predictions. It is important to recognize that these predictions (interpolations and extrapolations) are based upon patterns that have a degree of uncertainty.

The quality of the interpretation is directly related to the quality of the data. An awareness of uncertainty allows students to assess the reliability of data and data interpretation.

Chance addresses the predictability of the occurrence of an outcome. As students develop their understanding of probability, the language of mathematics becomes more specific and describes the degree of uncertainty more accurately.

## Essential Graduation Learnings

Essential graduation learnings are statements describing the knowledge, skills and attitudes expected of all students who graduate from high school. Essential graduation learnings are cross-curricular in nature and comprise different areas of learning: *aesthetic expression, citizenship, communication, personal development, problem solving, technological competence and spiritual and moral development.*

### Aesthetic Expression

*Graduates will be able to respond with critical awareness to various forms of the arts and be able to express themselves through the arts.*

### Citizenship

*Graduates will be able to assess social, cultural, economic and environmental interdependence in a local and global context.*

### Communication

*Graduates will be able to use the listening, viewing, speaking, reading and writing modes of language(s) and mathematical and scientific concepts and symbols to think, learn and communicate effectively.*

### Personal Development

*Graduates will be able to continue to learn and to pursue an active, healthy lifestyle.*

### Problem Solving

*Graduates will be able to use the strategies and processes needed to solve a wide variety of problems, including those requiring language and mathematical and scientific concepts.*



## Technological Competence

*Graduates will be able to use a variety of technologies, demonstrate an understanding of technological applications, and apply appropriate technologies for solving problems.*

## Spiritual and Moral Development

*Graduates will be able to demonstrate an understanding and appreciation for the place of belief systems in shaping the development of moral values and ethical conduct.*

See *Foundations for the Atlantic Canada Mathematics Curriculum*, pages 4-6.

The mathematics curriculum is designed to make a significant contribution towards students' meeting each of the essential graduation learnings (EGLs), with the communication, problem-solving and technological competence EGLs relating particularly well to the mathematical processes.

## Outcomes and Achievement Indicators

The curriculum is stated in terms of general outcomes, specific outcomes and achievement indicators.

### *General Outcomes*

**General outcomes** are overarching statements about what students are expected to learn in each course.

### *Specific Outcomes*

**Specific outcomes** are statements that identify the specific skills, understanding and knowledge that students are required to attain by the end of a given course.

In the specific outcomes, the word *including* indicates that any ensuing items must be addressed to fully meet the learning outcome. The phrase *such as* indicates that the ensuing items are provided for illustrative purposes or clarification, and are not requirements that must be addressed to fully meet the learning outcome.

### *Achievement Indicators*

**Achievement indicators** are samples of how students may demonstrate their achievement of the goals of a specific outcome. The range of samples provided is meant to reflect the scope of the specific outcome.

Specific curriculum outcomes represent the means by which students work toward accomplishing the general curriculum outcomes and ultimately, the essential graduation learnings.

## Program Organization

Program Level	Course 1	Course 2	Course 3	Course 4
Advanced	Mathematics 1201	Mathematics 2200	Mathematics 3200	Mathematics 3208
Academic		Mathematics 2201	Mathematics 3201	
Applied	Mathematics 1202	Mathematics 2202	Mathematics 3202	

The applied program is designed to provide students with the mathematical understandings and critical-thinking skills identified for entry into the majority of trades and for direct entry into the workforce.

The academic and advanced programs are designed to provide students with the mathematical understandings and critical-thinking skills identified for entry into post-secondary programs. Students who complete the advanced program will be better prepared for programs that require the study of calculus.

The programs aim to prepare students to make connections between mathematics and its applications and to become numerate adults, using mathematics to contribute to society.

## Summary

The conceptual framework for Grades 10-12 Mathematics (p.3) describes the nature of mathematics, mathematical processes and the mathematical concepts to be addressed. The components are not meant to stand alone. Activities that take place in the mathematics classroom should result from a problem-solving approach, be based on mathematical processes and lead students to an understanding of the nature of mathematics through specific knowledge, skills and attitudes among and between topics.

# ASSESSMENT AND EVALUATION

## Purposes of Assessment

What learning is assessed and evaluated, how it is assessed and evaluated, and how results are communicated send clear messages to students and others about what is really valued.

*Assessment* techniques are used to gather information for evaluation. Information gathered through assessment helps teachers determine students' strengths and needs in their achievement of mathematics and guides future instructional approaches.

Teachers are encouraged to be flexible in assessing the learning success of all students and to seek diverse ways in which students might demonstrate what they know and are able to do.

*Evaluation* involves the weighing of the assessment information against a standard in order to make an evaluation or judgment about student achievement.

Assessment has three interrelated purposes:

- assessment *for* learning to guide and inform instruction;
- assessment *as* learning to involve students in self-assessment and setting goals for their own learning; and
- assessment *of* learning to make judgements about student performance in relation to curriculum outcomes.

## Assessment **for** Learning

Assessment *for* learning involves frequent, interactive assessments designed to make student understanding visible. This enables teachers to identify learning needs and adjust teaching accordingly. It is an ongoing process of teaching and learning.

Assessment *for* learning:

- requires the collection of data from a range of assessments as investigative tools to find out as much as possible about what students know
- provides descriptive, specific and instructive feedback to students and parents regarding the next stage of learning
- actively engages students in their own learning as they assess themselves and understand how to improve performance.

**Assessment *as* Learning**

Assessment *as* learning actively involves students' reflection on their learning and monitoring of their own progress. It focuses on the role of the student as the critical connector between assessment and learning, thereby developing and supporting metacognition in students.

Assessment *as* learning:

- supports students in critically analysing their learning related to learning outcomes
- prompts students to consider how they can continue to improve their learning
- enables students to use information gathered to make adaptations to their learning processes and to develop new understandings.

**Assessment *of* Learning**

Assessment *of* learning involves strategies to confirm what students know, demonstrate whether or not they have met curriculum outcomes, or to certify proficiency and make decisions about students' future learning needs. Assessment *of* learning occurs at the end of a learning experience that contributes directly to reported results.

Traditionally, teachers relied on this type of assessment to make judgments about student performance by measuring learning after the fact and then reporting it to others. Used in conjunction with the other assessment processes previously outlined, however, assessment *of* learning is strengthened.

Assessment *of* learning:

- provides opportunities to report evidence to date of student achievement in relation to learning outcomes, to parents/guardians and other stakeholders
- confirms what students know and can do
- occurs at the end of a learning experience using a variety of tools.

Because the consequences of assessment *of* learning are often far-reaching, teachers have the responsibility of reporting student learning accurately and fairly, based on evidence obtained from a variety of contexts and applications.

## Assessment Strategies

Assessment techniques should match the style of learning and instruction employed. Several options are suggested in this curriculum guide from which teachers may choose, depending on the curriculum outcomes, the class and school/district policies.

### Observation (formal or informal)

This technique provides a way of gathering information fairly quickly while a lesson is in progress. When used formally, the student(s) would be aware of the observation and the criteria being assessed. Informally, it could be a frequent, but brief, check on a given criterion. Observation may offer information about the participation level of a student for a given task, use of a concrete model or application of a given process. The results may be recorded in the form of checklists, rating scales or brief written notes. It is important to plan in order that specific criteria are identified, suitable recording forms are ready, and all students are observed within a reasonable period of time.

### Performance

This curriculum encourages learning through active participation. Many of the curriculum outcomes promote skills and their applications. In order for students to appreciate the importance of skill development, it is important that assessment provide feedback on the various skills. These may be the correct manner in which to use a manipulative, the ability to interpret and follow instructions, or to research, organize and present information. Assessing performance is most often achieved through observing the process.

### Paper and Pencil

These techniques can be formative or summative. Whether as part of learning, or a final statement, students should know the expectations for the exercise and how it will be assessed. Written assignments and tests can be used to assess knowledge, understanding and application of concepts. They are less successful at assessing processes and attitudes. The purpose of the assessment should determine what form of paper and pencil exercise is used.

### Journal

Journals provide an opportunity for students to express thoughts and ideas in a reflective way. By recording feelings, perceptions of success, and responses to new concepts, a student may be helped to identify his or her most effective learning style. Knowing how to learn in an effective way is powerful information. Journal entries also give indicators of developing attitudes to mathematical concepts, processes and skills, and how these may be applied in the context of society. Self-assessment, through a journal, permits a student to consider strengths and weaknesses, attitudes, interests and new ideas. Developing patterns may help in career decisions and choices of further study.

**Interview**

This curriculum promotes understanding and applying mathematics concepts. Interviewing a student allows the teacher to confirm that learning has taken place beyond simple factual recall. Discussion allows a student to display an ability to use information and clarify understanding. Interviews may be a brief discussion between teacher and student or they may be more extensive. Such conferences allow students to be proactive in displaying understanding. It is helpful for students to know which criteria will be used to assess formal interviews. This assessment technique provides an opportunity to students whose verbal presentation skills are stronger than their written skills.

**Presentation**

The curriculum includes outcomes that require students to analyze and interpret information, to be able to work in teams, and to communicate information. These activities are best displayed and assessed through presentations. These can be given orally, in written/pictorial form, by project summary, or by using electronic systems such as video or computer software. Whatever the level of complexity, or format used, it is important to consider the curriculum outcomes as a guide to assessing the presentation. The outcomes indicate the process, concepts and context for which a presentation is made.

**Portfolio**

Portfolios offer another option for assessing student progress in meeting curriculum outcomes over a more extended period of time. This form of assessment allows the student to be central to the process. There are decisions about the portfolio, and its contents, which can be made by the student. What is placed in the portfolio, the criteria for selection, how the portfolio is used, how and where it is stored, and how it is evaluated are some of the questions to consider when planning to collect and display student work in this way. The portfolio should provide a long-term record of growth in learning and skills. This record of growth is important for individual reflection and self-assessment, but it is also important to share with others. For all students, it is exciting to review a portfolio and see the record of development over time.

## INSTRUCTIONAL FOCUS

### Planning for Instruction

Consider the following when planning for instruction:

- Integration of the mathematical processes within each topic is expected.
- By decreasing emphasis on rote calculation, drill and practice, and the size of numbers used in paper and pencil calculations, more time is available for concept development.
- Problem solving, reasoning and connections are vital to increasing mathematical fluency and must be integrated throughout the program.
- There should be a balance among mental mathematics and estimation, paper and pencil exercises, and the use of technology, including calculators and computers. Concepts should be introduced using manipulatives and be developed concretely, pictorially and symbolically.
- Students bring a diversity of learning styles and cultural backgrounds to the classroom. They will be at varying developmental stages.

### Teaching Sequence

The curriculum guide for Academic Mathematics 3201 is organized by units. This is only a suggested teaching order for the course. There are a number of combinations of sequences that would be appropriate.

Each two page spread lists the topic, general outcome, and specific outcome.

### Instruction Time Per Unit

The suggested number of hours of instruction per unit is listed in the guide at the beginning of each unit. The number of suggested hours includes time for completing assessment activities, reviewing and evaluating. The timelines at the beginning of each unit are provided to assist in planning. The use of these timelines is not mandatory. However, it is mandatory that all outcomes are taught during the school year, so a long term plan is advised. Teaching of the outcomes is ongoing, and may be revisited as necessary.

### Resources

The authorized resource for Newfoundland and Labrador students and teachers is *Principles of Mathematics 12* (Nelson). Column four of the curriculum guide references *Principles of Mathematics 12* for this reason. Teachers may use any other resource, or combination of resources, to meet the required specific outcomes.

## GENERAL AND SPECIFIC OUTCOMES

### GENERAL AND SPECIFIC OUTCOMES WITH ACHIEVEMENT INDICATORS (pages 19-228)

This section presents general and specific outcomes with corresponding achievement indicators and is organized by unit. The list of indicators contained in this section is not intended to be exhaustive but rather to provide teachers with examples of evidence of understanding that may be used to determine whether or not students have achieved a given specific outcome. Teachers may use any number of these indicators or choose to use other indicators as evidence that the desired learning has been achieved. Achievement indicators should also help teachers form a clear picture of the intent and scope of each specific outcome.

Academic Mathematics 3201 is organized into nine units: *Set Theory, Counting Methods, Probability, Rational Expressions and Equations, Polynomial Functions, Exponential Functions, Logarithmic Functions, Sinusoidal Functions and Financial Mathematics: Borrowing Money.*



# Set Theory

Suggested Time: 10 Hours

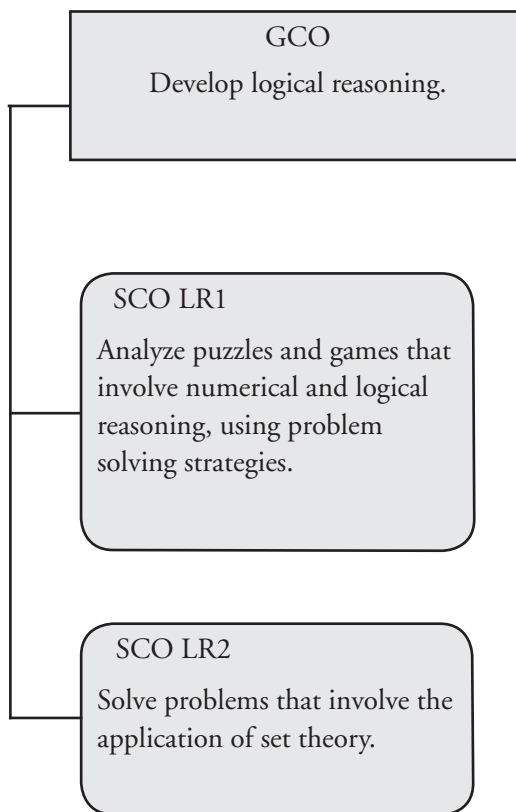
# Unit Overview

## Focus and Context

Students will be introduced to the notation, terminology and concepts related to Venn diagrams and set notation. Venn diagrams are commonly used to illustrate set intersections, unions, complements and subsets. Students will determine the number of elements in each region and then apply a formula or use a Venn diagram to determine the number of elements in the union of two or three sets.

Students will discover practical applications of set theory such as conducting Internet searches and solving puzzles.

## Outcomes Framework



## Mathematical Processes

[C] Communication	[PS] Problem Solving
[CN] Connections	[R] Reasoning
[ME] Mental Mathematics and Estimation	[T] Technology
	[V] Visualization

## SCO Continuum

Mathematics 1201	Mathematics 2201	Mathematics 3201
Algebra and Number	Number and Logic	Logical Reasoning
AN2 Demonstrate an understanding of irrational numbers by: <ul style="list-style-type: none"> <li>representing, identifying and simplifying irrational numbers</li> <li>ordering irrational numbers.</li> </ul> [CN, ME, R, V]	NL2 Analyze puzzles and games that involve spatial reasoning, using problem solving strategies. [CN, PS, R, V]	LR1 Analyze puzzles and games that involve numerical and logical reasoning, using problem solving strategies. [CN, ME, PS, R]  LR2 Solve problems that involve the application of set theory. [CN, PS, R, V]
Relations and Functions		
RF1 Interpret and explain the relationships among data, graphs, and situations. [C, CN, R, T, V]	not addressed	not addressed

## Logical Reasoning

### Outcomes

*Students will be expected to*

**LR2 Solve problems that involve the application of set theory.**

[CN, PS, R, V]

### Achievement Indicator:

LR2.1 *Provide examples of the empty set, disjoint sets, subsets and universal sets in context, and explain the reasoning.*

### Elaborations—Strategies for Learning and Teaching

In this unit, students organize information into sets and subsets. They use Venn diagrams to illustrate relationships between sets and subsets and use set notation to describe sets. Students determine the number of elements in each region of a Venn diagram and then determine the number of elements in the union of two or three sets.

Sets are a mathematical way to represent a collection or a group of objects, called elements. Students often deal with a set of books, for example, or a collection of hockey cards. The number of elements in a set  $A$  is denoted by  $n(A)$ .

In Mathematics 1201, students worked with domain and range using set notation and interval notation (RF1). There are situations where students will write algebraic expressions to describe sets. Provide them with a written description, such as, set  $M$  consists of the multiples of 3 from 1 to 100. Ask students to first rewrite the set as a list and then progress to an algebraic expression.

$$M = \{3, 6, 9, \dots, 93, 96, 99\}.$$

$$M = 3x, \{x \mid 1 \leq x \leq 33, x \in N\}.$$

To promote discussion, ask students the following questions:

- Is one form more efficient than the other?
- What does  $1 \leq x \leq 33$  represent?
- Why is it important to state that  $x$  belongs to the natural numbers?

Students should be introduced to the following types of sets: universal set, subset, empty set, disjoint set and complement of a set. Present students with an example similar to the following:

The universal set is defined as  $A$ , the set of natural numbers, and  $B$  is the set of natural numbers from 1 to 5.

Ask students the following questions to initiate discussion around the types of sets:

- Is it possible to explicitly list all the elements of both sets?
- Which set is finite and which set is infinite?
- Is one set a subset of the other?
- Which elements of the universal set do not belong to the subset?
- What is the complement set of  $B$ , defined as  $B'$ ?

Students should conclude that  $B$  is a subset of  $A$ , written as  $B \subset A$ . The complement set  $B'$  represents the natural numbers greater than 5.

Introduce students to examples of the empty set, such as the set of months with 32 days or the set of squares with five sides. Ask them to create their own empty sets and share their responses with the class.

**General Outcome: Develop logical reasoning.****Suggested Assessment Strategies***Interview*

- Ask students to explain whether a set can be considered a subset of itself.

(LR2.1)

- Mary created the sets  $P = \{1, 3, 4\}$  and  $Q = \{2, 3, 4, 5, 6\}$ . John stated that  $P \subset Q$  since the elements 3 and 4 are in both sets. Ask students if they agree or disagree with John and to explain their reasoning.

(LR2.1)

**Resources/Notes****Authorized Resource***Principles of Mathematics 12***1.1 Types of Sets and Set Notation**

Student Book (SB): pp. 6 - 18

Teacher Resource (TR): pp. 7 - 14

## Logical Reasoning

### Outcomes

*Students will be expected to*

LR2 Continued...

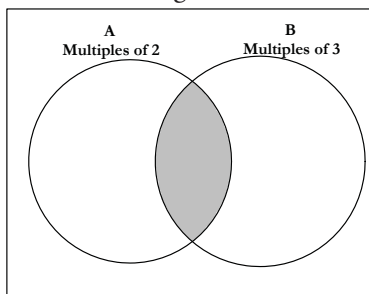
#### Achievement Indicators:

LR2.2 *Organize information such as collected data and number properties, using graphic organizers, and explain the reasoning.*

LR2.1 *Continued*

### Elaborations—Strategies for Learning and Teaching

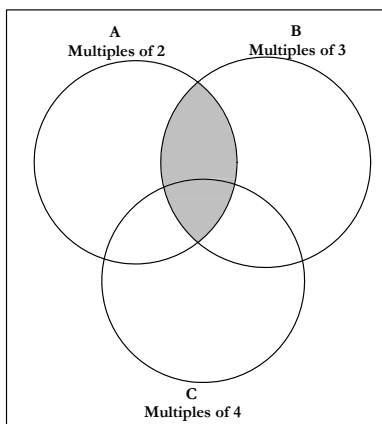
In Mathematics 1201, students were introduced to Venn diagrams to organize the various subsets of the real number system (AN2). In this unit, students continue to organize information into sets and subsets using Venn diagrams and explore what the different regions of the Venn diagram represent. Provide students with a set such as the following  $S = \{4, 5, 6, 8, 9, 11, 15, 17, 20, 24, 30, 32\}$  and the categories for its associated Venn diagram.



As students place the numbers in the appropriate location, they should consider:

- the universal set and how many elements are in this universal set
- the elements of each subset
- why the two circles overlap
- the common elements in set A and set B
- why some numbers are not placed in either circle

Ask students to add another circle to the Venn diagram to represent multiples of 4. They should think about where they should place the circle and whether the three circles overlap.



Discuss with students that there are situations where two (or more) sets may have no elements in common, known as disjoint sets. Ask students to draw a Venn diagram, for example, for the multiples of 3 and the factors of 8. Inform students that the two sets are defined as mutually exclusive.

**General Outcome: Develop logical reasoning.****Suggested Assessment Strategies***Paper and Pencil*

- Consider the following sets:  
 U is the universal set of playing cards in a standard 52-card deck,  
 S is the set of all 13 spades,  
 B is the set of all 26 black cards (spades and clubs),  
 D is the set of all 13 diamonds.

Ask students to respond to the following:

- Which of these sets are subsets of other sets?
- List the disjoint sets, if there are any.
- Represent the sets using a Venn diagram.

(LR2.1, LR2.2)

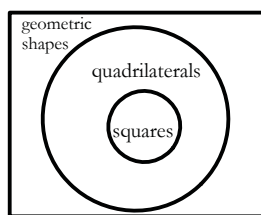
- Ask students to create their own disjoint sets and subsets. They should draw a Venn diagram for each situation. Consider the following samples to use as examples:

- C = set of consonants in the alphabet  
 V = set of vowels in the alphabet
- D = the set of prime divisors of 34  
 E = {2, 17}

(LR2.1, LR2.2)

*Interview*

- Provide each group with a Venn diagram illustrating various sets. Ask them to explain the different regions of the Venn diagram using the terms sets, subsets and disjoint. A sample is shown below.



Teachers should observe the groups and ask questions such as:

- Why is the set of squares a subset of the set of quadrilaterals?  
 Are there other subsets?
- What does the region outside of the quadrilateral circle represent?

(LR2.1, LR2.2)

**Resources/Notes****Authorized Resource**

*Principles of Mathematics 12*

**1.1 Types of Sets and Set Notation**

SB: pp. 6 - 18

TR: pp. 7 - 14

## Logical Reasoning

### Outcomes

*Students will be expected to*

LR2 Continued...

#### Achievement Indicators:

LR2.3 Explain what a specified region in a Venn diagram represents, using connecting words (*and, or, not*) or set notation.

LR2.4 Determine the elements in the complement, the intersection or the union of two sets.

LR2.2 Continued

### Elaborations – Strategies for Learning and Teaching

Students should continue to explore the different regions of a Venn diagram and be introduced to the intersection and union of sets using symbols and words. The intersection of set A and set B, for example, written as  $A \cap B$ , consists of all the elements that are common to both sets.

Provide students with a table where the set notation and its associated definition is provided. Ask students to draw the Venn diagram and write the elements that correspond to the notation.

- The universal set is the set of all integers from -3 to +3. Set A is the set of non-negative integers and set B is the set of integers divisible by 2.

Set Notation	Meaning	Venn Diagram	Answer
$A \cup B$ (A union B)	any element that is in either of the sets		{-2, 0, 1, 2, 3}
$A \cap B$ (A intersect B)	only elements that are in both A and B		{0,2}
$A \setminus B$ set A minus set B	elements found in set A but excluding the ones that are also in set B		{1,3}



**General Outcome: Develop logical reasoning.****Suggested Assessment Strategies***Paper and Pencil*

- R is the set of positive odd numbers less than 10. S is the set of multiples of 3 between 4 and 20. T is the set of prime numbers less than 12. Ask students to answer the following:
  - (i) List the elements of:
    - (a)  $R \cup S$
    - (b)  $R \cap S$
  - (ii) What does it mean to write  $x \in (R \cap T)$ ? List all possible values of  $x$ .
  - (iii) Is it true that  $S \cap T$  is the empty set? Explain your answer.

(LR2.2, LR2.3, LR2.4)

**Resources/Notes****Authorized Resource***Principles of Mathematics 12***1.2 Exploring Relationships between Sets**

SB: pp. 19 - 21

TR: pp. 15 - 18

**1.3 Intersection and Union of Two Sets**

SB: pp. 22 - 35

TR: pp. 19 - 28

Logical Reasoning

Outcomes

Students will be expected to

LR2 Continued ...

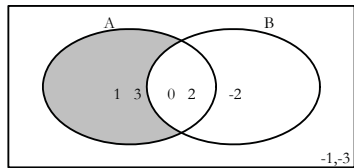
Achievement Indicators:

LR2.2, LR2.3, LR2.4  
Continued

Elaborations – Strategies for Learning and Teaching

Set Notation	Meaning	Venn Diagram	Answer
$A'$ (A complement or not A)	all elements in the universal set outside of A		$\{-3, -2, -1\}$
$(A \cup B)'$ not (A union B)	elements outside A and B		$\{-3, -1\}$
$(A \cap B)'$ not (A intersect B)	elements outside of the overlap of A and B		$\{-3, -2, -1, 1, 3\}$

Venn diagrams can help students develop formulas when determining the number of elements in certain sets. Ask students to predict a formula for  $n(A \setminus B)$ , the number of elements that are in set A but excluding the ones that are also in set B.



Some students may notice they can determine the number of elements in A and subtract the overlapping region,  $n(A \setminus B) = n(A) - n(A \cap B)$ . Others may determine the number of elements in both sets and subtract the number of elements in set B,  $n(A \setminus B) = n(A \cup B) - n(B)$ . Ask students if their formula works for disjoint sets. Students may choose to solve the problem using a formula or a Venn diagram to organize the information.

## General Outcome: Develop logical reasoning.

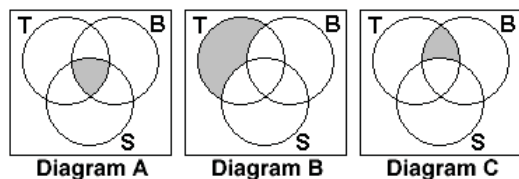
### Suggested Assessment Strategies

#### Performance

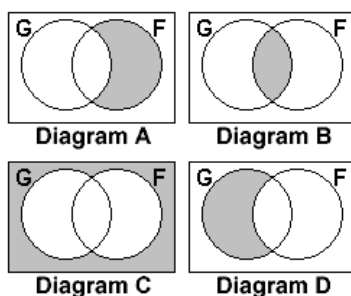
- Ask students to participate in the activity *Find your partner*. Half of the students should be given a card with a description on it and the other half should be given a card with a Venn diagram. Students need to move around the classroom to match the description with the correct Venn diagram. They should then present their findings to the class.

Consider the following samples shown below:

- (a) The diagrams below represent the activities chosen by youth club members. They can choose to play tennis (T), baseball (B) or swimming (S). Decide which diagram has the shading which represents the following descriptions:
- those who play all three sports
  - those who play tennis and baseball, but not swimming
  - those who play only tennis



- (b) The diagrams below represent a class of children. G is the set of girls and F is the set of children who like fencing. Decide which diagram has the shading which represents:
- girls who like fencing
  - girls who dislike fencing
  - boys who like fencing
  - boys who dislike fencing



(LR2.2, LR2.3, LR2.4)

### Resources/Notes

#### Authorized Resource

*Principles of Mathematics 12*

#### 1.2 Exploring Relationships between Sets

SB: pp. 19 - 21

TR: pp. 15 - 18

#### 1.3 Intersection and Union of Two Sets

SB: pp. 22 - 35

TR: pp. 19 - 28

## Logical Reasoning

### Outcomes

Students will be expected to

LR2 Continued...

#### Achievement Indicator:

LR2.5 Solve a contextual problem that involves sets, and record the solution, using set notation.

### Elaborations—Strategies for Learning and Teaching

Students should determine the number of elements in the union of two sets  $n(A \cup B)$ . Provide students with the following sets and ask them to predict the number of elements in the union of set A and B.

Set A has elements {2, 3, 6, 8, 9}

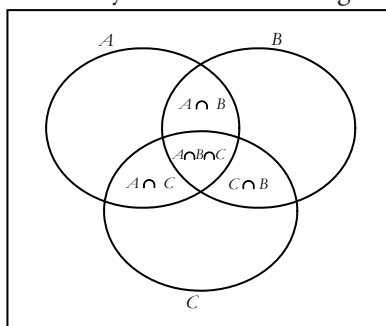
Set B has elements {4, 5, 6, 7, 9}

Using an algebraic approach, students may determine that  $n(A) = 5$  and  $n(B) = 5$  and initially make the assumption there are ten elements in the union of set A and B. Using a visual representation, such as a Venn diagram, should help them recognize that  $n(A \cup B) = 8$ . Use the following prompts to help students develop a formula for  $n(A \cup B)$ :

- Which elements were added twice when you added  $n(A)$  and  $n(B)$ ?
- How can you compensate for this over-counting?
- Predict a formula for  $n(A \cup B)$  using  $n(A)$ ,  $n(B)$  and  $n(A \cap B)$ .
- When will  $n(A \cup B) = n(A) + n(B)$ ?

When calculating the number of elements in the union of two sets, students should conclude that  $n(A \cup B) = n(A) + n(B) - n(A \cap B)$ . This result is known as the *Principle of Inclusion and Exclusion*. Ask students to determine the value of  $n(A \cap B)$  if the sets are disjoint. This mathematical principle informs students how to keep track of what to add and what to subtract in problems. It is important to give students a choice to either use a Venn diagram or the formula to solve problems involving union and intersections.

Students should analyze three intersecting sets.



Students should observe that there is one region in which all three sets overlap. There are three regions in which exactly two of the three sets overlap. When working with three intersecting sets, students continue to use the Principle of Inclusion and Exclusion to determine the number of elements in different sets. A formula could be used to determine  $n(A \cup B \cup C)$ . Due to the complexity of the formula, students are more likely to make errors, hence a Venn diagram would be a more efficient strategy for students.

**General Outcome: Develop logical reasoning.****Suggested Assessment Strategies***Paper and Pencil*

- A group of 30 students are surveyed to find out which of the three sports, soccer (S), basketball (B) or volleyball (V), they play. The results are as follows:

3 children do not play any of these sports

2 children play all three sports

6 play volleyball and basketball

3 play soccer and basketball

6 play soccer and volleyball

16 play basketball

12 play volleyball

When answering the following questions, it may be helpful to draw a Venn diagram to illustrate the relationship between the three sports played.

- (i) How many students play soccer only?
- (ii) How many students play soccer but not basketball?
- (iii) How many students play volleyball but not basketball?

(LR2.5)

**Resources/Notes****Authorized Resource**

*Principles of Mathematics 12*

**1.3 Intersection and Union of Two Sets**

SB: pp. 22 - 35

TR: pp. 19 - 28

## Logical Reasoning

### Outcomes

*Students will be expected to*

LR2 Continued...

### Achievement Indicator:

LR2.5 *Continued*

### Elaborations – Strategies for Learning and Teaching

A Venn diagram enables students to visualize the resulting set and easily account for the repetition of common elements. When students start filling out a three set Venn diagram, encourage them to start by completing the centre, then the two set intersections before filling the remainder of each set.

Students should solve problems such as the following:

There are 36 students who study science.

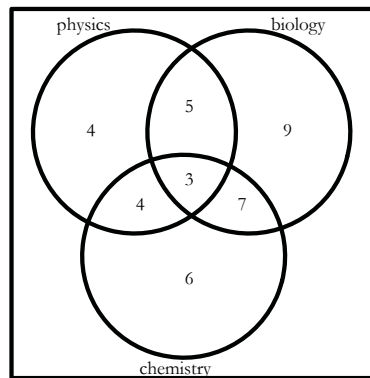
14 study physics, 18 study chemistry, 24 study Biology,

5 study physics and chemistry, 8 study physics and biology,

10 study biology and chemistry, 3 study all three subjects.

- (i) Determine the number of students who study Physics and Biology only.
- (ii) Determine the number of students who study at least two subjects.
- (iii) Determine the number of students who study biology only.

The relationships and patterns between sets can be explored using reasoning and a Venn diagram.



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**General Outcome: Develop logical reasoning.**

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**Suggested Assessment Strategies***Paper and Pencil*

- A survey of a machine shop reveals the following information about its employees: 44 employees can run a lathe, 49 employees can run the milling machine, 56 employees can operate a punch press, 27 employees can run a lathe and a milling machine, 19 employees can run a milling machine and operate a punch press, 24 employees can run a lathe and operate a press punch, 10 employees can operate all three machines, 9 employees cannot operate any of the three machines. Ask students to determine the number of people employed at the machine shop.

(LR2.5)

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**Resources/Notes****Authorized Resource***Principles of Mathematics 12***1.3 Intersection and Union of Two Sets**

SB: pp. 22 - 35

TR: pp. 19 - 28

## Logical Reasoning

### Outcomes

*Students will be expected to*

LR2 Continued...

#### Achievement Indicators:

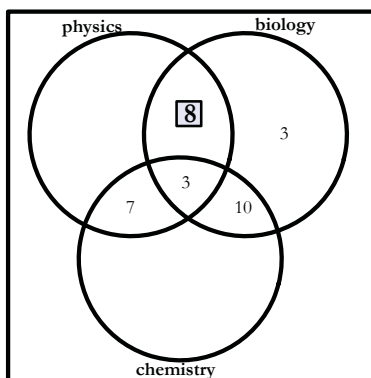
LR2.6 *Identify and correct errors in a solution to a problem that involves sets.*

LR2.7 *Explain how set theory is used in applications such as Internet searches, database queries, data analysis, games and puzzles.*

LR2.5 *Continued*

### Elaborations—Strategies for Learning and Teaching

A common student error occurs when students do not take into account the overlapping region of the three sets. If 8 people studied physics and biology, they sometimes write 8 instead of 5 forgetting that 3 people studied all three. Students may draw an incorrect Venn diagram such as the following:



When completing a Venn diagram, ask students to identify how the sets overlap as well as the region that satisfies neither of the sets. When these regions are completed, it may become easier for students to identify what the totals should be for each circle. Remind students that the numbers in all the regions total the number in the universal set.

The Internet is often used to gather information. An illustration of the use of set relationships would be the manner in which some search engines process searches. Students could, for example, search for topics that interest them and come up with ways where they can refine their search. They will connect words and phrases with "and" to search for sites that contain both. Students can google, for example, "cats and dogs" which shows all sites with the words cats and dogs represented by the notation  $C \cap D$ . If they search "cats or dogs", a search engine using this syntax will return all web pages containing either the word "cats" or the word "dogs" or both. This corresponds to the set  $C \cup D$ . Ask students which search would produce more results. Ask students what the search result would be if they googled "cats not dogs" and what the notation would be written as for this search. This application reinforces the mathematical model of a Venn diagram for their searches.



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**General Outcome: Develop logical reasoning.**


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**Suggested Assessment Strategies**
*Paper and Pencil*

- A firm manufactures three types of shampoo: Shine, Bubble, and Glory. 1000 families were surveyed and the results were posted in an advertisement as follows:

843 use Shine  
 673 use Bubble  
 585 use Glory  
 600 use both Shine and Bubble  
 423 use both Shine and Glory  
 322 use both Bubble and Glory  
 265 use all three types

Ask students identify where the error occurs in the survey results as reported in the advertisement. It may be helpful to illustrate the information using a Venn diagram.

(LR2.6)

*Performance*

- When performing a Web search, you are defining a set as the collection of those Web sites that have some common feature in which you are interested. There are three logical operators, AND, OR, and AND NOT, that help in narrowing your search to a manageable number of sites that might be most useful for a particular project. Ask students the following:
  - (i) What do you know about making effective searches for information on the Internet? How is this related to set theory?
  - (ii) In groups of two, conduct an Internet search on a topic of your choice. Record each successive search, including the number of hits (challenge students to obtain 100 hits or fewer). Display student results around the classroom.

(LR2.5, LR2.7)

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**Resources/Notes**
**Authorized Resource***Principles of Mathematics 12***1.3 Intersection and Union of Two Sets**

SB: pp. 22 - 35

TR: pp. 19 - 28

## Logical Reasoning

### Outcomes

*Students will be expected to*

LR2 Continued...

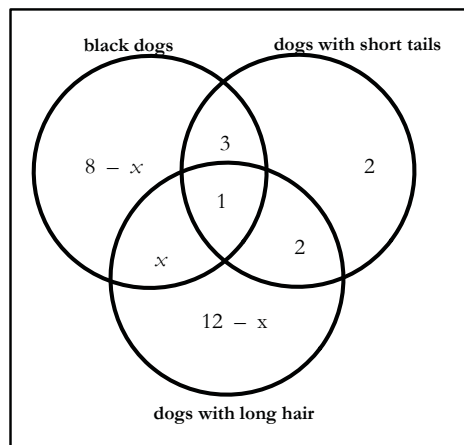
### Achievement Indicators:

LR2.5, LR2.7 *Continued*

### Elaborations—Strategies for Learning and Teaching

There are problems involving sets where students should use an equation to solve the problem. They should use the clues or information to define the sets, identify regions of the Venn diagram, and identify the region of interest using a variable. Ask students to solve a problem such as the following:

- There are twenty five dogs at the dog show. Twelve of the dogs are black, eight of the dogs have short tails, and fifteen of the dogs have long hair. There is only one dog that is black with a short tail and long hair. Three of the dogs are black with short tails and do not have long hair. Two of the dogs have short tails and long hair but are not black. If all of the dogs in the kennel have at least one of the mentioned characteristics, how many dogs are black with long hair but do not have short tails?



Solving the equation,  $(8 - x) + (x) + (4) + (4) + (12 - x) = 25$ , students should determine there are 3 dogs that are black with long hair.

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**General Outcome: Develop logical reasoning.**


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**Suggested Assessment Strategies**
*Paper and Pencil*

- 40 members in a sports club were surveyed:
  - 2 play all three sports
  - 23 play ball hockey
  - 24 play tennis
  - 18 play golf
  - 14 play tennis and ball hockey
  - 8 play tennis and golf
  - 1 member makes the refreshments and does not play any sport

Ask students to determine the number of people who play ball hockey and golf.

(LR2.5, LR2.7)

- In a survey of 55 people, the following results were recorded:
  - 13 people like Hawaiian pizza
  - 19 people like pepperoni pizza
  - 26 people like cheese pizza
  - 15 people do not like pizza
  - 5 people like Hawaiian pizza and pepperoni pizza, but not cheese pizza
  - 2 people likes all types of pizza
  - 2 people like Hawaiian pizza and cheese pizza, but not pepperoni pizza

Ask students to determine how many people like only cheese pizza. It may be helpful to illustrate the information using a Venn diagram.

(LR2.5, LR2.7)

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**Resources/Notes**
**Authorized Resource**

*Principles of Mathematics 12*

**1.4 Applications of Set Theory**

SB: pp. 39 - 54

TR: pp. 30 - 44

## Logical Reasoning

### Outcomes

*Students will be expected to*

**LR1 Analyze puzzles and games that involve numerical and logical reasoning, using problem-solving strategies.**

[C, CN, ME, PS, R]

#### Achievement Indicators:

LR1.1 *Determine, explain and verify a strategy to solve a puzzle or to win a game; e.g.,*

- *guess and check*
- *look for a pattern*
- *make a systematic list*
- *draw or model*
- *eliminate possibilities*
- *simplify the original problem*
- *work backward*
- *develop alternative approaches.*

LR1.2 *Identify and correct errors in a solution to a puzzle or in a strategy for winning a game.*

### Elaborations—Strategies for Learning and Teaching

In Mathematics 2201, students applied problem-solving strategies to analyze puzzles and games that involved spatial reasoning. This outcome is intended to be integrated throughout the course by using puzzles and games focused on numerical and logical reasoning.

Students need time to play and enjoy the game before analysis begins. They can then discuss the game, determine the winning strategies and explain these strategies through demonstration, oral explanation or in writing.

A variety of puzzles and games, such as board games, online puzzles and games, appropriate selections for gaming systems, and pencil and paper games should be used. It is not intended that the activities be taught in a block of time, but rather explored periodically during the year. Timing and integration of this outcome should be included in teacher planning throughout the course. Therefore, students can be exposed to three or four games at different times, whether it be at the beginning or end of each unit, or a set “game day”. Students could engage in a game when they are finished other work. As students work through the different games and puzzles, they will begin to develop effective strategies for solving the puzzle or game.

Games provide opportunities for building self-concept, enhancing reasoning and decision making, and developing positive attitudes towards mathematics through reducing the fear of failure and error. In comparison to more formal activities, greater learning can occur through games due to increased interaction between students, opportunities to explore intuitive ideas and problem-solving strategies. Students’ thinking often becomes apparent through the actions and decisions they make during a game, so teachers have the opportunity to formatively assess learning in a non-threatening situation.

Problem-solving strategies will vary depending on the puzzle or game. Some students will explain their strategy by working backwards, looking for a pattern, using guess and check, or by eliminating possibilities, while others will plot their moves by trying to anticipate their opponents’ moves. As students play games and analyze strategies, they explore mathematical ideas and compare different strategies for efficiency.

There may be situations where students are able to play the game and solve problems but are unable to determine a winning strategy. Teachers could participate with the group and think through the strategies out loud so the group can hear the reasoning for selected moves. Ask the groups’ opinions about moves in the game and facilitate discussions around each of the other players’ moves and strategies.

**General Outcome: Develop logical reasoning.****Suggested Assessment Strategies***Observation*

- Playing games creates dialogue and provides a tool for informal assessment. Stations could be set up with one or two games at each centre. Teachers should circulate among the groups and assess each student's understanding. Puzzles and games involving numerical and logical reasoning could include:

- |               |                    |
|---------------|--------------------|
| (i) Chess     | (v) Sudoku™        |
| (ii) Nim™     | (vi) Magic squares |
| (iii) Kakuro™ | (vii) Cribbage     |
| (iv) Yahtzee™ |                    |

Students can work through the puzzles on their own either individually or with a partner. They could record their progress on a sheet such as the one shown below.

Puzzle	Solved	Strategy	Comments/Hints

(LR1.1)

*Journal*

- Ask students to write about the puzzles and games they found interesting and why.

(LR1.1)

*Performance*

- Using a game or puzzle of their choice, ask students to write their own description of the game/puzzle, the rules of play, and helpful hints. They should give the information to another classmate as they play the game.
- Students can play *Fast Figuring*. Using the number of cards from an ordinary deck, five cards are dealt to each player. One more card is turned up to reveal the target number. Players race to use their five cards and any four operations (+, −, × ÷) to form a statement that results in the target number. The first player to do so wins a point. If, after 3 minutes, no one can find a solution, the players show their hands for checking, then the cards are shuffled and play continues.

(LR1.1)

**Resources/Notes****Authorized Resource***Principles of Mathematics 12***1.4 Applications of Set Theory**

SB: pp. 39 - 54

TR: pp. 30 - 44

**Web Link**

[www.rinkworks.com/brainfood/c/logic.shtml](http://www.rinkworks.com/brainfood/c/logic.shtml)

Brain Food contains Mathematical Reasoning Puzzles that require both logical and mathematical reasoning, as well as number Puzzles such as Scrambled Equations.

## Logical Reasoning

### Outcomes

*Students will be expected to*

LR1 Continued...

### Achievement Indicators:

LR1.1, LR1.2 *Continued*

### Elaborations – Strategies for Learning and Teaching

The following are some tips for using games in the mathematics class:

- Use games for specific purposes.
- Keep the number of players between two to four so that turns come around quickly.
- Communicate to students the purpose of the game.
- Engage students in post-game discussions.

As students play a game, it is important to pose questions and engage them in discussions about the strategies they are using.

When introducing games, students will need to understand the rules and procedures of the game. Consider the following:

- Introduce the game to one group of students while others are completing individual work. Then divide the whole class into groups, putting a student from the first group into each of the other groups to teach them the game.
- Choose students to play the game as a demonstration, possibly with assistance in decision making from the whole class.
- Divide the class into groups. Play the game with the whole class, with each group acting as a single player.

When working with puzzles and games, teachers could set up learning stations. Consider the following tips when creating the stations:

- Depending on the nature of the game, some stations may require multiple games, while another station may involve one game that requires more time to play.
- Divide students into small groups. At regular intervals, have students rotate to the next station.

## General Outcome: Develop logical reasoning.

### Suggested Assessment Strategies

#### Discussion

- As students play games or solve puzzles, ask probing questions and listen to their responses. Record the different strategies and use these strategies to begin a class discussion. Possible discussion starters include:
  - (i) Thumbs up if you liked the game, thumbs sideways if it was okay, and thumbs down if you didn't like it. What did you like about it? Why?
  - (ii) What did you notice while playing the game?
  - (iii) Did you make any choices while playing?
  - (iv) Did anyone figure out a way to quickly find a solution?

(LR1.1)

#### Performance

- In *Roll Six*, players roll six dice and use five of the numbers together with any four operations to make the sixth number. Points are scored for successful equations.

(LR1.1)

#### Observation

- Ask students to take any three-digit number whose digits are not all the same. Arrange the digits in decreasing order, and then rearrange them in increasing order and subtract the two three-digit numbers. With the resulting three digit solution, repeat the process. If the difference consists of two digits, use a 0 if necessary. Consider the example using digits 2, 7, and 9:

$$\begin{array}{r}
 \underline{972} \\
 - 279 \\
 \hline
 693
 \end{array}
 \quad
 \begin{array}{r}
 \underline{963} \\
 - 369 \\
 \hline
 594
 \end{array}
 \quad
 \begin{array}{r}
 \underline{954} \\
 - 459 \\
 \hline
 495
 \end{array}
 \quad
 \begin{array}{r}
 \underline{954} \\
 - 459 \\
 \hline
 495
 \end{array}$$

This process will lead to the number 495 again, known as Kaprekar number.

- (i) In groups of 2-3 students, ask each group to apply the process of Kaprekar to a two-digit number in which the digits are not the same (i.e., 9 is written as 09). Students should compare their results and discuss what they notice.
- (ii) Ask students to repeat the process for a four-digit number. They should compare their results and write a conjecture about the resulting number.

(LR1.1)

### Resources/Notes

#### Authorized Resource

*Principles of Mathematics 12*

#### 1.4 Applications of Set Theory

SB: pp. 39 - 54

TR: pp. 30 - 44

#### Web Link

An interactive numerical puzzle can be found at:

<http://www.goobix.com/games/numerical-puzzle/>

## Logical Reasoning

### Outcomes

*Students will be expected to*

LR1 Continued...

#### Achievement Indicators:

LR1.1, LR1.2 *Continued*

### Elaborations—Strategies for Learning and Teaching

As students work through the games and puzzles, teachers should keep a checklist of the games and puzzles each student is working on. This would be a good opportunity for students to work in groups where each has been exposed to a different game or puzzle. Ask the group of students to:

- Explain the rules of the game in their own words to other students.
- Give a brief demonstration.
- Explain the strategy they tried in solving the puzzle or playing the game.
- What general advice would they give to other students trying to solve the puzzle or play the game?
- What they did when they were having difficulty?

Invite students to bring in their own games and puzzles that involve spatial reasoning and have not been covered in class. This may involve bringing games or puzzles from home or searching the Internet to find games, puzzles or online applications of interest. Students will inform the rest of their class about the game/puzzle using questions such as the following:

- What is the puzzle/game and where did you find it?
- Describe the puzzle/game. Why did you select this puzzle/game?
- What is the object of the game and the rules of play?
- What strategy was used to solve the puzzle or win the game?

Many logic puzzles are based on statements which contain clues to the solution of the problem. These clues may be positive or negative (i.e., they may tell you part of the answer or tell you what the answer is not). Discuss with students that it is just as important to know what can be eliminated as it is to find the actual correct answer. To keep track of these possibilities (and non-possibilities), students could use tables, lists, or Venn diagrams; whichever is most helpful for the type of puzzle they are solving.



## General Outcome: Develop logical reasoning.

### Suggested Assessment Strategies

#### *Performance*

- Given a  $5 \times 5$  grid, ask students to arrange the numbers in order, from 1 to 25. The first four numbers, from 1 to 5, must be ordered on the first row, the next 5 numbers on the second row and so on. Ask students to follow the directions and answer the questions:
  - (i) Choose any number in the first row and circle it. Then cross out all the numbers in the same column below the circled number.
  - (ii) Circle any one of the remaining numbers in the second row. Then cross out all the numbers in the same column above and below the circled number.
  - (iii) Repeat this process for the third and fourth row.
  - (iv) Circle the remaining number in the fifth row and cross out all the numbers in the same column above the circled number. What is the sum of the circled numbers?
  - (v) Repeat steps (i) to (iv) by circling a different number in the first row. How does the sum of the circled numbers compare with your result above?
  - (vi) What is the sum of the circled numbers on a 3 by 3 grid? 4 by 4 grid?
  - (vii) Choose a grid of your choice and explain why the sum of the circled numbers is always the same.

(LR1.1)

- In a magic square, the sum of the numbers in each row, column, and diagonal is the same, in this case 15. Ask students to rearrange the numbers so that the sums for each row, column and diagonal are all different, and none of the sums is 15.

(LR1.1)

### Resources/Notes

#### Authorized Resource

*Principles of Mathematics 12*

#### 1.4 Applications of Set Theory

SB: pp. 39 - 54

TR: pp. 30 - 44

#### Web Link

Interactive logic puzzles can be found at:

<http://www.conceptispuzzles.com>

## Logical Reasoning

### Outcomes

Students will be expected to

LR1 Continued...

#### Achievement Indicators:

LR1.1, LR1.2 *Continued*

### Elaborations – Strategies for Learning and Teaching

Ask students to work through the following clues to solve a problem such as the following:

- Four friends met on Saturday morning for breakfast. Each friend ordered a different drink and breakfast meal during their visit and when it was time to leave, each got a different drink to go. Determine the first name of each friend, the drink and meal each ordered for breakfast, and the drink each ordered to go.
1. Brenda had waffles but not an espresso.
  2. The friend who ordered the pancakes also ordered decaf coffee to go but didn't have cranberry juice.
  3. The woman who ordered the omelet had water to drink but she wasn't Amy.
  4. The two friends who ordered juice were Emily and the friend who ordered an egg sandwich.
  5. The friend who ordered a cappuccino didn't order orange juice.
  6. Melony ordered a hot tea to go.

Encourage students to draw an organized list similar to the following:

	cranberry juice	milk	orange juice	water	egg sandwich	omelet	pancakes	waffles	cappuccino	decaf coffee	espresso	hot tea
Amy												
Brenda												
Emily												
Melony												
cappuccino												
decaf coffee												
espresso												
hot tea												
egg sandwich												
omelet												
pancakes												
waffles												

Place an X in a box when you can eliminate that possibility. Place an O in a box which you know is correct.

Students can then draw a solution table:

First Name	Breakfast Drink	Breakfast Meal	Drink to Go
Amy			
Brenda			
Emily			
Melony			

**General Outcome: Develop logical reasoning.****Suggested Assessment Strategies***Paper and Pencil*

- Ask students to use the clues to solve the puzzle:
  - (i) Amanda, Kim, Alex and Sarah each have different coloured cars. One car is red, one blue, one white and the other is black.

Clue 1 : Amanda's car is not red or white.

Clue 2 : Kim's car is not blue or white.

Clue 3 : Alex's car is not black or blue.

Clue 4 : Sarah's car is red.

Find out which person has which car.

(LR1.1)

- Three containers can hold 19 L, 13 L and 7 L of water, respectively. The 19-L container is empty. The other two are full. Ask students how they can measure 8L of water using no other container and no other water supply.

(LR1.1)

**Resources/Notes****Authorized Resource**

*Principles of Mathematics 12*

**1.4 Applications of Set Theory**

SB: pp. 39 - 54

TR: pp. 30 - 44

## Logical Reasoning

### Outcomes

*Students will be expected to*

LR1 Continued...

#### Achievement Indicator:

LR1.3 *Create a variation on a puzzle or a game, and describe a strategy for solving the puzzle or winning the game.*

### Elaborations – Strategies for Learning and Teaching

As an alternative, students may have an idea for a game or puzzle that would challenge their classmates. To create a game, they could use the rules of an existing game, but use different materials or add extra materials. They could also use the idea for a game and change the rules. Another option is to use a board game and add math tasks to it. Rather than writing tasks directly onto the boards, they can place coloured stickers on certain spaces and make up colour-coded cards with questions. A game such as Snakes and Ladders™, for example, can be modified to Operations Snakes and Ladders. The board can be used with two dice. On each turn, to determine the number of spaces to move, the player has the option of multiplying, dividing, adding or subtracting the two numbers, with a maximum answer of twenty.

The following guiding questions could be used to help students evaluate their games:

- Can the game be completed in a short time?
- Is there an element of chance built in?
- Are there strategies which can be developed to improve the likelihood of winning?

When working with a KenKen puzzle, for example, students can create their own puzzles, exchange with a partner, solve, and then discuss the solution. Use the following guidelines when playing the puzzle:

- choose a grid size
- fill in the numbers from 1 to the grid size
- do not repeat a number in any row or column
- the numbers in each heavily outlined set of squares (called cages) must combine (in any order) to produce the target number in the top corner using the mathematical operation indicated
- cages with one square should be filled in with the target number in the top corner
- a number can be repeated within a cage as long as it is not in the same row or column

1	2÷		6×
2÷		3	
5+	3-		
	3	5+	

5+ 2	3+ 1	3 3
3 3	2 2	1- 1
3÷ 1	3 3	2 2

**General Outcome: Develop logical reasoning.****Suggested Assessment Strategies***Performance*

- In a magic square, the sum of the numbers in each row, column, and diagonal is the same. Ask students to arrange the numbers 1 through 9 on a tic-tac-toe board such that the numbers in each row, column, and diagonal add up to 15.

Note: answers may vary depending whether the digits are distinct or repeated.

6	1	8
7	5	3
2	9	4

1	9	5
9	5	1
5	1	9

(LR1.3)

*Paper and Pencil*

- Jane is playing a game of Sudoku. Ask students to help her determine the value of A and B.

		5				6	2	7
	3		4	5				
8	9		2					
		3			9	2		
	5		7		8		6	
		4				9	7	
4			5		6		A	B
5				9	2		3	6
3						5		2

(LR1.1)

**Resources/Notes****Authorized Resource***Principles of Mathematics 12***1.4 Applications of Set Theory**

SB: pp. 39 - 54

TR: pp. 30 - 44

**Web Link**

Kenken puzzles can be found at:  
[www.kenken.com](http://www.kenken.com)



# Counting Methods

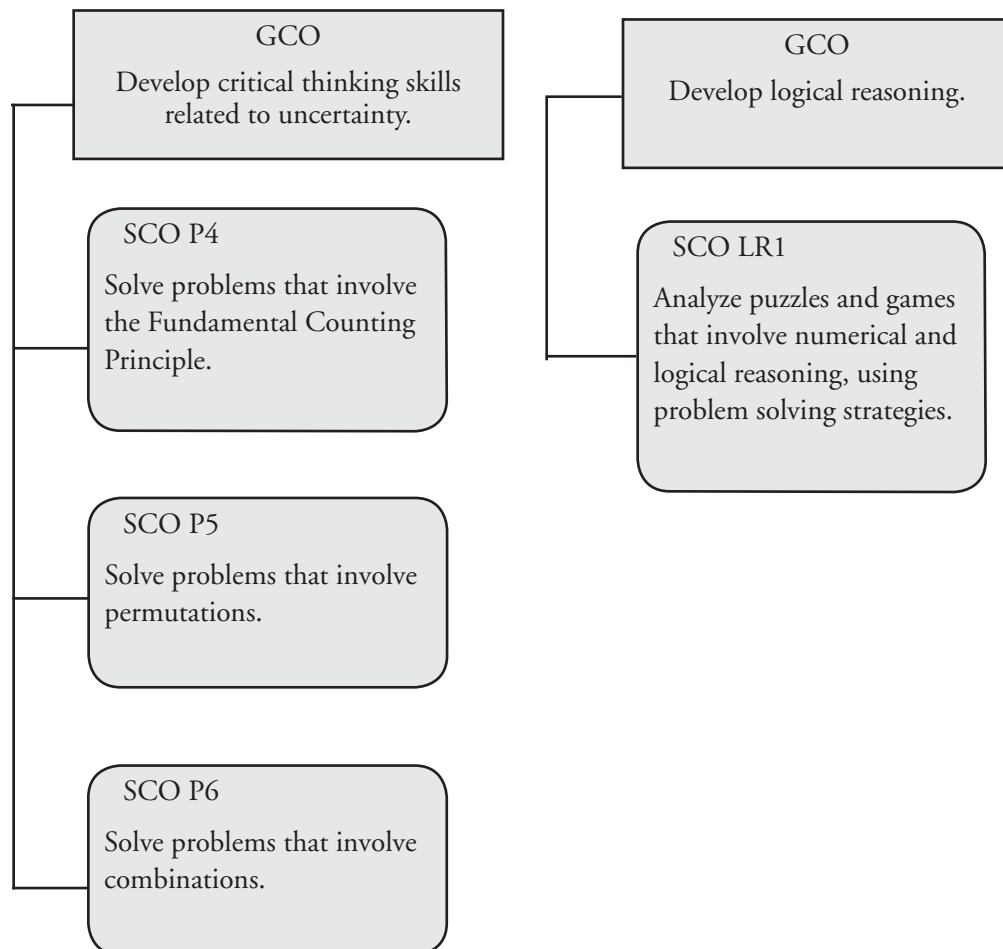
Suggested Time: 14 Hours

# Unit Overview

## Focus and Context

In this unit, students will solve counting problems that involve the Fundamental Counting Principle, permutations and combinations. When working with permutations, they will make the connection to factorial notation using the Fundamental Counting Principle. Students will decide which strategy to use to solve a counting problem by determining if order is important. They will also consider conditions, repetition, and if a problem involves multiple tasks.

## Outcomes Framework





## Mathematical Processes

[C] Communication  
 [CN] Connections  
 [ME] Mental Mathematics  
 and Estimation

[PS] Problem Solving  
 [R] Reasoning  
 [T] Technology  
 [V] Visualization

## SCO Continuum

Mathematics 1201	Mathematics 2201	Mathematics 3201
Probability		
not addressed	not addressed	<p>P4 Solve problems that involve the fundamental counting principle.            [PS, R, V]</p> <p>P5 Solve problems that involve permutations.            [ME, PS, R, T, V]</p> <p>P6 Solve problems that involve combinations.            [ME, PS, R, T, V]</p>
Number and Logic		Logical Reasoning
not addressed	<p>NL2 Analyze puzzles and games that involve spatial reasoning, using problem solving strategies.            [CN, PS, R, V]</p>	<p>LR1 Analyze puzzles and games that involve numerical and logical reasoning, using problem solving strategies.            [CN, ME, PS, R]</p>

## Probability

### Outcomes

*Students will be expected to*

**P4 Solve problems that involve the Fundamental Counting Principle.**

[PS, R, V]

#### Achievement Indicators:

P4.1 *Represent and solve counting problems, using a graphic organizer.*

P4.2 *Generalize, using inductive reasoning, the Fundamental Counting Principle.*

P4.3 *Identify and explain assumptions made in solving a counting problem.*

P4.4 *Solve a contextual counting problem, using the Fundamental Counting Principle, and explain the reasoning.*

### Elaborations—Strategies for Learning and Teaching

The Fundamental Counting Principle is a means to find the number of ways of performing two or more operations together. Students should develop an understanding of how the principle works, when it can be used, and the advantages and disadvantages it has over methods that involve direct counting, when used to solve counting problems. Later, the Fundamental Counting Principle will be applied in work with permutations and combinations.

Tree diagrams were used in grade 7 (7SP5 and 7SP6) and Grade 8 (8SP2) to determine the number of possible outcomes in probability problems. The counting principle is all about choices students make when given many possibilities. The following example could be used to activate students' prior knowledge.

- The school cafeteria restaurant offers a lunch combo for \$6 where a person can order 1 sandwich (chicken, turkey, grilled cheese), 1 side (fruit, yogurt, soup) and 1 drink (juice, milk).

Ask students to draw a tree diagram or create a table to determine the possible lunch combos.

Modifying this example to include a fourth category or adding more options in one of the categories can be used to illustrate the limitations on the practicality of using graphic organizers for counting problems and offers a good introduction to the Fundamental Counting Principle. When the sample space is very large the task of listing and counting all the outcomes in a given situation is unrealistic. The Fundamental Counting Principle enables students to find the number of outcomes without listing and counting each one.

In the previous example, there were **three** sandwich choices. For each of these, there are **three** side choices and for each side choice there are **two** beverage choices. Multiplying the number of options from each category gives possible meal choices. This should agree with their results from the tree diagram. This illustrates the Fundamental Counting Principle, which states, if there are  $r$  ways to perform a task,  $s$  ways to perform a second task,  $t$  ways to perform a third task, etc., then the number of ways of performing all the tasks together is  $r \times s \times t \dots$

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**General Outcome: Develop critical thinking skills related to uncertainty.**


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**Suggested Assessment Strategies**
*Observation*

- In groups of two or three students, ask students to discuss the advantages and disadvantages of using graphic organizers to determine the sample space compared to using the Fundamental Counting Principle.

Teachers should observe the groups and ask questions such as the following:

- Which solution do you prefer?
- Why do you prefer that solution?

(P4.1, P4.2)

*Paper and Pencil*

- Ask students to answer the following:
  - How many variations of ice cream sundaes can be made if the ice cream choices are vanilla and chocolate, the toppings are nuts or candy pieces, and the sauces are chocolate, fudge or strawberry. Each sundae must contain one type of ice cream, one topping, and one sauce. They should draw a tree diagram to show how many different types of sundaes can be made and then verify the solution using the Fundamental Counting Principle.
 

(P4.1, P4.3, P4.4)
  - In Newfoundland and Labrador a license plate consists of a letterletter-letter-digit-digit-digit arrangement such as CXT 132.
    - How many license plate arrangements are possible?
    - How many license plate arrangements are possible if no letter or digit can be repeated?
    - How many license plate arrangements are possible if vowels (a, e, i, o, u) are not allowed?
 

(P4.3, P4.4)
  - Canadian postal codes consist of a letter-digit-letter-digit-letter-digit arrangement.
    - How many codes are possible, and how does this compare with the number of license plates in Newfoundland and Labrador?
    - In Newfoundland and Labrador, all postal codes begin with the letter A. How many postal codes are possible?
 

(P4.3, P4.4)

**Resources/Notes**
**Authorized Resource**

*Principles of Mathematics 12*

**2.1 Counting Principles**

Student Book (SB): pp. 66-75

Teacher Resource (TR): pp. 72-78

## Probability

### Outcomes

*Students will be expected to*

P4 Continued...

#### Achievement Indicators:

P4.1, P4.3, P4.4 *Continued*

### Elaborations—Strategies for Learning and Teaching

Students previously calculated the number of possibilities when choosing a sandwich and a side and a beverage. When working through examples, it is also important for students to distinguish between the word "and" and "or". To help make this comparison, ask students to calculate the number of possibilities when choosing a sandwich or a side or a beverage. Ask them what effect the word "or" has on their answer.

They should conclude that when choosing a sandwich and a side and a beverage; the word "and" indicates these three selections (operations) are performed together, so the number of ways of doing each individual selection are multiplied. If, instead, a sandwich or a side or a beverage is being selected, then students should result in 8 possibilities (3 sandwich choices + 3 side choices + 2 beverage choices).

The intent of this discussion is instructional in nature and is meant to help students understand why, using the Fundamental Counting Principle, the individual operations are multiplied rather than added.

Use examples that are relevant to the student. Ask students to determine, for example, the total number of ways to fill the six spaces on a license plate, where the first three spaces are letters and the last three spaces are numbers. Using the Fundamental Counting Principle, students should recognize there are 17 576 000 possibilities ( $26 \times 26 \times 26 \times 10 \times 10 \times 10$ ). Ask students how this answer would change if repetition of letters was not allowed. Initiate a discussion by asking them how a province could increase the total number of possible license plates. One response may be to make the plates with four letters and two numbers (45 697 600 possibilities).

In groups of three, students could participate in the following activity. The first student creates restrictions for a license plate. The second student determines the number of ways to fill the spaces on the license plate. The third student verifies the solution.

**General Outcome: Develop critical thinking skills related to uncertainty.****Suggested Assessment Strategies***Journal*

- Ask students to respond to the following:

Greg is trying to select a new cell phone based on the following categories :

Brands: Ace, Best, Cutest

Colour: Lime, Magenta, Navy, Orange

Plans: Text, Unlimited Calling

With the aid of a tree diagram, a table, and/or organized list, explain why it makes sense to multiply the options from each category to determine the number of ways of selecting Greg's new cell phone.

(P4.1, P4.2, P4.4)

*Presentation*

- Ask students to research the format and restrictions on a license plate in a province or country of their choice (i.e., some letters are not used, such as I, O and/or Q to avoid confusion with the digits 1 and 0). They should determine the number of plates possible and present their finding to the class. Students could try to convince the class their system is the best. The class could debate the merits of each system presented.

(P4.3, P4.4)

**Resources/Notes****Authorized Resource**

*Principles of Mathematics 12*

**2.1 Counting Principles**

Student Book (SB): pp. 66-75

Teacher Resource (TR): pp. 72-78

## Probability

### Outcomes

*Students will be expected to*

P4 Continued...

#### Achievement Indicators:

P4.1, P4.3, P4.4 *Continued*

### Elaborations—Strategies for Learning and Teaching

Provide students with the following examples to help them differentiate between the Fundamental Counting Principle and the Principle of Inclusion and Exclusion (LR2). They should understand the mathematical meaning behind the words "and" and "or", and the strategy they will use to solve problems that involve these words.

- How many possible outcomes exist if we first flip the coin and then roll the die?
- How many possible outcomes exist if you either flip the coin or roll the die?
- Determine the number of ways that, on a single die, the result could be odd or greater than 4?
- A buffet offers 5 different salads, 10 different entrees, 8 different desserts and 6 different beverages. In how many different ways can you choose a salad, an entree, a dessert, and a beverage?

Students should recognize that "and" implies multiplication and the use of the Fundamental Counting Principle. The word "or" implies addition and the use of Principle of Inclusion and Exclusion, depending on whether the tasks are mutually exclusive or not.

Problems involving the Fundamental Counting Principle sometimes contain restrictions. When arranging items, for example, a particular position must be occupied by a particular item. Students should be exposed to examples such as the following where restrictions exist:

- In how many ways can a teacher seat five boys and three girls in a row of eight seats if a girl must be seated at each end of the row?

Teachers should ask the following questions to help guide students as they work through the example:

- (i) Are there any restrictions for seating girls and boys?
- (ii) Why should you fill the girls seats first?
- (iii) How many choices are there for seat 1 if a girl must sit in that seat?
- (iv) How many girls remain to sit in seat 8?
- (v) How many choices of boys and girls remain to sit in each of seats 2 through 7?
- (vi) Which mathematical operation should you use to determine the total number of arrangements?

Students should conclude, by the Fundamental Counting Principle, the teacher can arrange the girls and boys in  $3 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 \times 2 = 4320$  ways.

**General Outcome: Develop critical thinking skills related to uncertainty.****Suggested Assessment Strategies***Paper and Pencil*

- Ask students to answer the following:
  - (i) How many three-digit numbers can you make using the digits 1, 2, 3, 4 and 5? Repetition of digits is not allowed.
  - (ii) How does the application of the Fundamental Counting Principle change if repetition of the digits is allowed? Determine how many three-digit numbers can be formed that include repetitions.
  - (iii) How many ways can you order the letters MUSIC if it must start with a consonant and end with a vowel?

(P4.3, P4.4)

*Presentation*

- Ask students to create and solve their own problems that require the use of the Fundamental Counting Principle. They should exchange problems and compare their solutions.

(P4.4)

**Resources/Notes****Authorized Resource***Principles of Mathematics 12***2.1 Counting Principles**

SB: pp. 66 - 75

TR: pp. 72 - 78

## Probability

### Outcomes

*Students will be expected to*

**P5 Solve problems that involve permutations.**

[ME, PS, R, T, V]

#### Achievement Indicators:

P5.1 *Represent the number of arrangements of  $n$  elements taken  $n$  at a time, using factorial notation.*

P5.2 *Determine, with or without technology, the value of a factorial.*

### Elaborations—Strategies for Learning and Teaching

Students will be introduced to factorial notation and how this relates to the concept of permutations. A formula will be developed and applied in problem solving situations, including those that involve permutations with conditions. Students should first be introduced to permutations of  $n$  different elements taken  $n$  at a time. They will then move to permutations of  $n$  different elements taken  $r$  at a time.

Ask students to think about examples where numbers must be used in a specific sequence (i.e., the order of the numbers is important). One suggestion may be a student lock since they are often secured by a 3-number arrangement.

Permutations is an ordered arrangement of all or part of a set. Provide students some time to determine the possible permutations of the letters A, B, and C. They should write ABC, ACB, BAC, BCA, CAB, CBA as their list. When discussing this example, it may be beneficial to compare a permutation with a combination. Ask students how many combinations are possible. Ask them why the number of combinations is less than the number of permutations. Students should recognize the arrangement ABC and BCA are the same combination since order does not matter, resulting in 4 combinations ABC, ABD, ACD, and BCD. The concept of combinations will be explored in greater detail in the next section.

When working with permutations, it is important for students to make the connection to the Fundamental Counting Principle and to factorial notation  $n!$ . In the previous example, there were three letters (A, B and C) in the arrangement to consider. For the first letter, students should notice there are three options since any of the three letters can be selected. Each time a letter is placed, there is one fewer letter to choose from the group. As they write the product of the numbers ( $3 \times 2 \times 1 = 6$ ), they should observe that this process is similar to the Fundamental Counting Principle. Introduce this product in abbreviated form as  $3!$ .

Continue to provide students with examples where they determine the number of ways to arrange  $n$  distinct objects. Ask students to determine the number of ways to arrange or permute a group of five people in a line. They should fill in the table provided.

1 <sup>st</sup> person	2 <sup>nd</sup> person	3 <sup>rd</sup> person	4 <sup>th</sup> person	5 <sup>th</sup> person
5 options	4 options remaining	3 options remaining	2 options remaining	1 option remaining

By the Fundamental Counting Principle, there are  $5 \times 4 \times 3 \times 2 \times 1$  or 120 ways or  $5!$ . This should allow them to make the conjecture that the number of permutations is  $n!$ .



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**General Outcome: Develop critical thinking skills related to uncertainty.**

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**Suggested Assessment Strategies***Interview*

- Ask students to create a permutation problem that can be solved by evaluating  $8!$ . They should exchange their problems with a partner to discuss the similarities and differences.

(P5.1, P5.2)

*Paper and Pencil*

- In how many different ways can a set of 5 distinct books be arranged on a shelf?
- In how many different orders can 15 different people stand in a line?

(P5.1, P5.2)

**Resources/Notes****Authorized Resource***Principles of Mathematics 12***2.2 Introducing Permutations and Factorial Notation**

SB: pp. 76 - 83

TR: pp. 79 - 83

## Probability

### Outcomes

*Students will be expected to*

P5 Continued...

#### Achievement Indicators:

P5.1, P5.2 *Continued*

P5.3 *Simplify a numeric or an algebraic fraction that contains factorials in both the numerator and the denominator.*

P5.4 *Solve an equation that involves factorials.*

### Elaborations—Strategies for Learning and Teaching

Introduce students to factorial notation as a more concise way of representing the product of consecutive descending natural numbers. Generally,  $n! = n(n - 1)(n - 2)(n - 3) \dots (2)(1)$  where  $n \in \mathbb{N}$ . Note that  $0!$  will be addressed in the context of permutations after the formula for  ${}_n P_r$  is introduced.

Ask students to complete the following table and comment on any patterns they notice.

$n$	$n!$	$n(n - 1)!$
1	1	1
2	2 x 1	2 x 1! = 2
3	3 x 2 x 1	3 x 2! = 6
4	4 x 3 x 2 x 1	4 x 3! = 24
5	5 x 4 x 3 x 2 x 1	5 x 4! = 120

They should observe that  $n! = n(n - 1)!$ . Ask students to use the value of  $5!$ , for example, to determine  $6!$ .

In preparation for working with formulas involving permutations, students should be given opportunities to simplify factorial expressions. Ask students to simplify an expression such as  $\frac{9!}{6!}$ . Initially, they may rewrite each factorial in its expanded form and then cancel common factors to obtain the final answer. Ask them to think about a more efficient way to rewrite the numerator so they do not have to expand the denominator. Students sometimes mistakenly reduce  $\frac{9!}{6!} = \frac{3!}{2!}$ . Encourage them to write out the expanded form so they can identify their mistake.

Students should simplify expressions where the product can be expressed as a single factorial or the expression can be written as the product of two binomials. Ask students to solve problems similar to  $(n + 1)(n)!$  and  $\frac{n!}{(n-2)!}$ .

Students should also solve algebraic equations that involve factorial notation. In some cases, extraneous roots will occur due to the nature of the definition of factorial notation. When students solve the equation  $\frac{(n)!}{(n-2)!} = 20$ , for example, ask them if both solutions  $n = 5$  and  $n = -4$  are correct and to explain why or why not. Since factorial notation is only valid for natural numbers, students should realize that  $n = -4$  is an extraneous root. Limit examples where the resulting equation is the product of two binomials resulting in a quadratic equation. Students should solve and then verify their solutions.

Students are not expected to solve an equation such as  $\frac{(n)!}{(n-3)!} = 1716$  where the resulting equation  $n(n-1)(n-2) = 1716$  is cubic. They are only familiar with solving quadratic equations.

**General Outcome: Develop critical thinking skills related to uncertainty.**

### Suggested Assessment Strategies

#### *Paper and Pencil*

- Sheila is attempting to simplify  $\frac{640!}{638!4!}$ . She wrote the following steps:

$$\begin{aligned} & \frac{640!}{638!4!} \\ & \frac{640 \times 639 \times 638!}{638!4!} \\ & \frac{640 \times 639}{4!} \\ & 160 \frac{640 \times 639}{4!} \\ & 102\,240 \end{aligned}$$

Ask students to explain the strategy that was used in the second step of the solution. They should then proceed to identify and explain any errors in the students work. Ask students to write the correct solution.

(P5.3)

- Ask students to solve the following equations for  $n$ :

$$\begin{aligned} \text{(i)} \quad & \frac{(n+2)!}{(n+1)!} = 10 \\ \text{(ii)} \quad & \frac{(n+5)!}{(n+3)!} = 56 \\ \text{(iii)} \quad & \frac{(n)!}{(n-2)!} = 182 \\ \text{(iv)} \quad & \frac{2(n+3)!}{(n+1)!} = 180 \end{aligned}$$

(P5.3, P5.4)

### Resources/Notes

#### Authorized Resource

*Principles of Mathematics 12*

#### 2.2 Introducing Permutations and Factorial Notation

SB: pp. 76 - 83

TR: pp. 79 - 83

## Probability

### Outcomes

Students will be expected to

P5 Continued...

#### Achievement Indicators:

P5.5 Determine the number of permutations of  $n$  elements taken  $r$  at a time.

P5.6 Generalize strategies for determining the number of permutations of  $n$  elements taken  $r$  at a time.

### Elaborations—Strategies for Learning and Teaching

Students should also work with problems where only some of the objects are used in each arrangement (i.e., arranging a subset of items). Students can use the Fundamental Counting Principle to develop an understanding of a permutation of  $n$  elements taken  $r$  at a time,  ${}_n P_r = \frac{n!}{(n-r)!}$ ,  $0 \leq r \leq n$ . This formula should make more sense if students can see how it is derived from the work they have already completed with factorials. Students can use the following example to help them with the derivation.

- How many ways can a set of 5 students be selected for the positions of class president, vice-president and secretary?

Using the Fundamental Counting Principle, students should write  $5 \times 4 \times 3$ . Their earlier exposure to simplifying factorial expressions should help them realize this expression can be rewritten as  $\frac{5!}{2!}$ . Assist students to make the connection between this value and the formula  $\frac{5!}{(5-3)!} = {}_5 P_3$ .

When working with the permutation formula, it is important for students to use factorial notation and the cancellation of common factors. Using the formula to evaluate  ${}_6 P_4$ , for example, encourage students to use mental math to simplify  $\frac{6!}{2!} = \frac{6 \times 5 \times 4 \times 3 \times 2!}{2!}$ . When  $n$  and  $r$  are large, however, using the formula in conjunction with a calculator would be more manageable.

Emphasize to students that in the notation,  ${}_n P_r$ ,  $n$  is the number of elements in a set and  $r$  is the number of elements to be selected at any given time. Ask students can the value of  $r$  be bigger than  $n$ . Ask them if it is possible to select more elements than are actually in the set. When using the formula,  ${}_n P_r = \frac{n!}{(n-r)!}$ , if  $r$  is greater than  $n$ , students should conclude the denominator would contain a factorial of a negative number, which is undefined.

Ask students how the formula changes if all of the objects are used in the arrangement. Using substitution where  $n = r$ , results in  ${}_n P_n$  which is  $n!$ . If the number of permutations of six people arranged in a line is  $6!$ , ask students to illustrate this using the permutation formula. They should be able to explain that this example is a permutation of a set of 6 objects from a set of 6. Therefore, applying the formula  ${}_n P_r = \frac{n!}{(n-r)!}$ , the result is  ${}_6 P_6 = \frac{6!}{(6-6)!} = \frac{6!}{0!}$ . This means  $6! = \frac{6!}{0!}$ , and the only value of  $0!$  that makes sense is  $0! = 1$ .

Evaluating  $0!$  means determining the number of ways there are to count an empty set. Since there is nothing to count, ask students *In how many ways can one count nothing?* A mathematical answer to this is one.

**General Outcome: Develop critical thinking skills related to uncertainty.**

### Suggested Assessment Strategies

#### Observation

- Ask students to arrange a set of four different coloured blocks into as many different configurations into groups of two blocks. They should explicitly state any assumptions they are making and then calculate the number of configurations to verify their results.

(P5.5)

#### Paper and Pencil

- Ask students to answer the following:
  - There are 10 movies playing at a theatre. In how many ways can you see two of them consecutively?
  - A soccer league has 12 teams, and each team plays each other twice; once at home, and once away. How many games are scheduled?

(P5.5)

- Ask students to make a conjecture about the value of  $0!$  using the following pattern:

$4! = 24$	
$3! = 6$	$3! = \frac{4!}{4}$
$2! = 2$	$2! = \frac{3!}{3}$
$1! = 1$	$1! = \frac{2!}{2}$
$0! = ?$	$0! = \frac{1!}{1}$

(P5.5)

#### Journal

- A code consists of three letters chosen from A to Z and three digits chosen from 0 to 9, with no repetition of letters or numbers. Students should explain why the total number of possible codes can be found using the expression  ${}_{26}P_3 \times {}_{10}P_3$ .

(P5.5)

### Resources/Notes

#### Authorized Resource

*Principles of Mathematics 12*

#### 2.3 Permutations When All Objects Are Distinguishable

SB: pp. 84 - 95

TR: pp. 84 - 91

## Probability

### Outcomes

*Students will be expected to*

P5 Continued...

#### Achievement Indicators:

P5.5, P5.6 *Continued*

### Elaborations – Strategies for Learning and Teaching

Students should solve problems where arrangements are created with and without repetition. Consider the arrangement of a 5-digit password if only the digits 0-9 can be used. Ask students the following:

- Is the order of the digits in a password important? Explain.
- How many arrangements are possible if repetition is allowed? Do the number of choices stay the same or are they reduced?
- How many arrangements are possible if repetition is not allowed? Do the number of choices stay the same or are they reduced?
- In which case is there a greater number of permutations possible?

Students should make the following conclusions:

- If repetition is allowed, the number of arrangements will be  $10 \times 10 \times 10 \times 10 \times 10 = 10^5$ .
- If replacement or repetition is not allowed, then the same password would have  $10 \times 9 \times 8 \times 7 \times 6 = {}_{10}P_5$  possibilities.

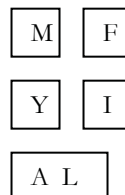
To be successful, it is important for students to read the problem carefully and decide whether they should apply the Fundamental Counting Principle or use the permutation formula. Ask students to work through a mixture of problems. They should be asking themselves the following questions:

- Does the Fundamental Counting Principle apply?
- Is it possible to use the formula  ${}_nP_r$  ?

Permutation problems sometimes involve conditions. In certain situations, objects may be arranged in a line where two or more objects must be placed together, or certain objects(s) must be placed in certain positions.

- How many arrangements of the word FAMILY exist if A and L must always be together?

When certain items are to be kept together, students should treat the joined items as if they were only one object.



Discuss with students there are 5 groups in total, and they can be arranged in  $5!$  ways. The letters AL can be arranged in  $2!$  ways. The total arrangements would be  $5! \times 2! = 240$ . This is a good opportunity to ask students why they used the Fundamental Counting Principle to multiply the two conditions.

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**General Outcome: Develop critical thinking skills related to uncertainty.**


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**Suggested Assessment Strategies**
*Paper and Pencil*

- Ask students to answer the following:
  - (i) The code for a lock consists of three numbers selected from 0, 1, 2, 3, with no repeats. For example, the code 1-2-1 would not be allowed but 3-0-2 would be allowed. Using the permutation formula, determine the number of possible codes.
  - (ii) How many two digit numbers can be formed using the digits 1, 2, 3, 4, 5, 6 if repetition is allowed?
  - (iii) How many distinct arrangements of three letters can be formed using the letters of the word LOCKERS?
  - (iii) How many 3 letter arrangements can be formed using the letters M, A, T allowing for repetition of the letters?

P(5.5)

- Jean, Kyle, Colin and Lori are to be arranged in a line from left to right.
  - (i) How many ways can they be arranged?
  - (ii) How many ways can they be arranged if Jean and Lori cannot be side by side?
  - (iii) How many ways can they be arranged if Kyle and Colin must be side by side?
  - (iv) How many ways can they be arranged if Jean must be at one end of the line?

P(5.5)

**Resources/Notes**
**Authorized Resource**
*Principles of Mathematics 12*
**2.3 Permutations When All Objects Are Distinguishable**

SB: pp. 84 - 95

TR: pp. 84 - 91

## Probability

### Outcomes

*Students will be expected to*

P5 Continued...

#### Achievement Indicators:

P5.5, P5.6 *Continued*

P5.4 *Continued*

### Elaborations – Strategies for Learning and Teaching

Students should work through counting problems that include words such as "at most" or "at least". Ask students to solve a problem such as the following:

- To open the garage door of Mary's house, she uses a keypad containing the digits 0 through 9. The password must be at least a 4 digit code to a maximum of 6 digits, and each digit can only be used once in the code. How many different codes are possible?

Guide students as they calculate the number of permutations possible for 10 numbers, where each arrangement uses 4, 5 or 6 of those objects. They should realize there are three cases to consider ( ${}_{10}P_4 + {}_{10}P_5 + {}_{10}P_6$ ).

Knowledge of permutations should be applied to solve equations of the form  ${}_nP_r = k$ . Students should solve equations where they simplify the expression so that it no longer contains factorials. Limit examples where they are only required to work with equations where they solve for  $n$ . They will not be expected to solve the equation for  $r$ .

Example 1  ${}_nP_2 = 30$

$$\frac{n!}{(n-2)!} = 30$$

$$n(n-1) = 30$$

- Students can reason that consecutive integers with a product of 30 are needed. Since  $6(6-1) = 30$ ,  $n = 6$ .
- Alternatively, they can solve the quadratic equation  $n^2 - n - 30 = 0$ . This equation has roots 6 and -5. Since  $n$  must be positive, -5 is extraneous.

Example 2  ${}_{n-1}P_2 = 12$

$$\frac{(n-1)!}{(n-1-2)!} = 12$$

$$\frac{(n-1)!}{(n-3)!} = 12$$

$$\frac{(n-1)(n-2)(n-3)!}{(n-3)!} = 12$$

$$(n-1)(n-2) = 12$$

- Students could reason that consecutive integers with a product of 12 will satisfy this equation. Therefore,  $n = 5$  because  $(5-1)(5-2) = 12$ .
- They could also solve the quadratic equation  $n^2 - 3n + 2 = 12$  to determine that  $n = 5$  (-2 is an extraneous root).



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**General Outcome: Develop critical thinking skills related to uncertainty.**

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**Suggested Assessment Strategies**

*Paper and Pencil*

- Ask students to solve the following:

(i)  ${}_n P_2 = 42$

(ii)  ${}_{n+1} P_2 = 20$

(P5.4)

- Mary has a set of posters to arrange on her bedroom wall. She can only fit two posters side by side. If there are 72 ways to choose and arrange two posters, how many posters does she have in total?

(P5.4)

**Resources/Notes**

**Authorized Resource**

*Principles of Mathematics 12*

**2.3 Permutations When All Objects Are Distinguishable**

SB: pp. 84 - 95

TR: pp. 84 - 91

**Note:**

Question 16 on p. 94 of the SB is not an expectation in this course.

## Probability

### Outcomes

*Students will be expected to*

P5 Continued...

#### Achievement Indicators:

P5.7 *Determine the number of permutations of  $n$  elements taken  $n$  at a time where some elements are not distinct.*

P5.8 *Explain, using examples, the effect on the total number of permutations of  $n$  elements when two or more elements are identical.*

### Elaborations—Strategies for Learning and Teaching

A permutation of  $n$  elements taken  $n$  at a time ( ${}_n P_n$  or  $n!$ ) is affected if one or more elements in the set are identical. If a set of 3 marbles consists of 2 identical green marbles and 1 blue marble, students may initially write the number of arrangements as  $3!$ . They may not realize that the set  $\{G1, G2, B\}$  is identical to  $\{G2, G1, B\}$ . This configuration is counted as two different arrangements instead of one, therefore, it must be removed from the total count. Students can eliminate the extraneous cases by dividing out repetitions ( $\frac{3!}{2!}$ ). A set of  $n$  objects containing  $a$  identical objects of one kind and  $b$  identical objects of another kind can be arranged in  $\frac{n!}{a!b!}$ . Dividing  $n!$  by  $a!$  and  $b!$  eliminates arrangements that are the same and that would otherwise be counted multiple times.

---

**General Outcome: Develop critical thinking skills related to uncertainty.**


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**Suggested Assessment Strategies**
*Paper and Pencil*

- Ask students to answer the following:
  - (i) If there are 9 different cookies (4 chocolate chip, 3 oatmeal and 2 raisin), in how many different orders can you eat all of them?
  - (ii) How many ways can the letters in ACTION be ordered if all the vowels must be kept together?
  - (iii) How many ways can you order the letters of BANANAS if it begins with "AA" where the third letter is not A?
  - (iv) How many ways can the letters MASK be formed?
  - (v) Determine the number of arrangements (of any number of letters) that can be formed from the word MASK? (Hint: arrangement can be two letters, three letters, four letters)
  - (vi) Show that you can form 120 distinct five-letter arrangements from GREAT but only 60 distinct five-letter arrangements from GREET.
  - (vii) How many distinct arrangements can be formed using all the letters of STATISTICS?

(P5.7, P5.8)

*Journal*

- Find the total number of arrangements of the word SILK and the total number of arrangements of the word SILL. How do your answers compare? Explain why this relationship exists.

(P5.7, P5.8)

**Resources/Notes**
**Authorized Resource**
*Principles of Mathematics 12*
**2.4 Permutations When Objects Are Identical**

SB: pp. 98 - 108

TR: pp. 92 - 97

## Probability

### Outcomes

*Students will be expected to*

**P6 Solve problems that involve combinations.**

[ME, PS, R, T, V]

#### Achievement Indicators:

P6.1 *Explain, using examples, why order is or is not important when solving problems that involve permutations or combinations.*

P6.2 *Determine the number of combinations of  $n$  elements taken  $r$  at a time.*

P6.3 *Generalize strategies for determining the number of combinations of  $n$  elements taken  $r$  at a time.*

### Elaborations—Strategies for Learning and Teaching

In contrast to permutations, combinations are an arrangement of objects without regard to order. A formula will be developed and applied in problem solving situations.

To distinguish between a permutation and a combination, students should be given a situation for each where the number of possibilities can be determined with simple counting methods. They have been introduced to permutations as an arrangement of objects in which order matters. Combinations, however, is a grouping of objects where order does not matter. Ask students to identify problems, such as the following, as a permutation or a combination.

- "My fruit salad is a combination of apples, grapes and strawberries." Students should recognize that whether it is a combination of "strawberries, grapes and apples" or "grapes, apples and strawberries", it is the same fruit salad.
- "The combination to the safe was 4-7-2". In this case, students should recognize that the order 7-2-4 or 2-4-7 would not work. It has to be exactly 4-7-2.

It is important for students to highlight the differences between permutations and combinations. Ask them to think about the following example:

- In a lottery, six numbers from 1 to 49 are selected (Lotto 6/49). A winning ticket must contain the same six numbers but they may be in any order.

Ask students the following:

- If order matters, determine the number of permutations.  
 $({}_{49}P_6 = \frac{49!}{(49-6)!} = 10\,068\,347\,520)$ .
- If order does not matter, why is it necessary to divide the value by 6! (the number of ways of arranging the six selected numbers)? The number of combinations is  
 ${}_{49}C_6 = \frac{49!}{(49-6)!6!} = \frac{49 \times 48 \times 47 \times 46 \times 45 \times 44}{6!} = 13\,983\,816$ .
- Ask students why the number of combinations is less than the number of permutations.

Communicate to students that generally, given a set of  $n$  objects taken  $r$  at a time, the number of possible combinations is  ${}_nC_r = \frac{n!}{r!(n-r)!}$ . Students should make the connection between permutations and combinations ( ${}_nC_r = \frac{{}_nP_r}{r!}$ ). Teachers could use the following example and prompts to investigate the combination formula:

- An assignment consists of three questions (A, B, C) and students are required to attempt two. Starting with an arrangement involving a small number will give students a visual representation of the possibilities.

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**General Outcome: Develop critical thinking skills related to uncertainty.**


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**Suggested Assessment Strategies**
*Paper and Pencil*

- Ask students to answer the following:
  - (i) At a local ice-cream store, you can order a sundae with a choice of toppings. There are three different sauces to choose from (chocolate, strawberry, butterscotch) and four different dry toppings (peanuts, smarties, M&M, sprinkles). When selecting one sauce and one dry topping, how many different sundaes could you order?
  - (ii) A volleyball team has 12 players. How many ways can the coach choose the starting line-up of 6 players?
  - (iii) If a committee of 8 people is to be formed from a pool of 13 people, but Mitchell and Lisa must be on the committee, how many selections can be made?

(P6.2, P6.3)

*Performance*

- Ask students to participate in a *Quiz-Quiz-Trade* activity. Provide students with cards, each having a scenario which is either a permutation or combination. Student 1 must read a card to student 2 who then decides whether it is a permutation or combination, explaining why they think so. They then switch roles, after which the students will trade cards and find another partner.

(P6.1)

**Resources/Notes**
**Authorized Resource**
*Principles of Mathematics 12*
**2.5 Exploring Combinations**

SB: pp. 109 - 110

TR: pp. 98 - 101

**2.6 Combinations**

SB: pp. 111- 120

TR: pp. 102 - 106

**Web Link**

The following site contains a permutations/combinations group activity.

<http://www.deltastate.edu>

keyword: combinations and permutations summer institute 2005

## Probability

### Outcomes

*Students will be expected to*

P6 Continued...

#### Achievement Indicators:

P6.2, P6.3 *Continued*

### Elaborations—Strategies for Learning and Teaching

- Calculate the number of permutations for choosing two of the three questions. ( ${}_3P_2 = \frac{3!}{(3-2)!} = 6$ )
- Write the number of arrangements to verify your answer. (AB, BA, BC, CB, AC, CA)
- How many ways can two questions be arranged? (2!)
- Is the order in which questions are chosen important? (selection AB is the same as BA)
- Why is it necessary to divide  ${}_3P_2$  by 2! to determine the number of combinations?

Discuss with students that the order in which questions are chosen is not important, therefore, each group of 2 permutations is just one combination. The number of combinations of  $n$  elements taken  $r$  at a time is represented by  ${}_nC_r = \frac{n!}{r!(n-r)!}$ . This may be denoted as  $\binom{n}{r}$ .

Similar to permutations, teachers should provide combination problems where students may have to use the Fundamental Counting Principle or problems that involve conditions. A good application type problem involves asking students to form sub-committees. The following examples illustrates the different algebraic manipulations students should perform when working with combinations.

- The student council decides to form a sub-committee of 5 members to plan their Christmas Concert. There are a total of 11 student members: 5 males and 6 females. Ask students to determine how many different ways the sub-committee can consist of exactly three females.

An understanding that the question is asking for a sub-committee of 3 females out of 6 and 2 males out of 5 is important here. Since order of the committee is not important, they will use combinations to solve the problem. This is represented by  ${}_6C_3 \times {}_5C_2$ .

- Ask students to determine how many different ways the sub-committee can consist of at least three females.

Ensure students are looking for a sub-committee of: 3 females and 2 males or 4 females and 1 male or 5 females and 0 males. This is represented by  $({}_6C_3 \times {}_5C_2) + ({}_6C_4 \times {}_5C_1) + ({}_6C_5 \times {}_5C_0)$ .

- Ask students to determine how many different ways the sub-committee can consist of at least one female.

Students should recognize that they are looking for a sub-committee of 1 female/4 males or 2 females/3 males or 3 females/2 males, or 4 females /1 male or 5 females/0 males. As an alternative, they could subtract the possibilities with 0 females and 5 males from the total number of combinations  $({}_5C_{11} - ({}_6C_0 \times {}_5C_5))$ .

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**General Outcome: Develop critical thinking skills related to uncertainty.**


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**Suggested Assessment Strategies**
*Observation*

- Ask students to create a display or foldable in which they list or draw all of the possible combinations for a scenario and verify the answer using a combination formula (i.e., a pizza shop offers 5 toppings). Ask them how many different three-topping pizzas can be made.

(P6.2, P6.3)

*Paper and Pencil*

- Ask students to answer the following:
  - From a standard deck of 52 cards, how many 5-card hands have:
    - at least one red card
    - at least one face card (J,K, or Q)
    - at most 3 aces
    - no spades?
  - A volleyball coach decides to use a starting line-up of 1 setter, 2 middle hitters, 2 power hitters and 1 right side hitter. She chooses 14 players for the team: 3 setters, 4 middle hitters, 4 power hitters, 3 right side hitters. How many possible starting line-ups are there?

(P6.2, P6.3)

**Resources/Notes**
**Authorized Resource**
*Principles of Mathematics 12*
**2.5 Exploring Combinations**

SB: pp. 109 - 110

TR: pp. 98 - 101

**2.6 Combinations**

SB: pp. 111 - 120

TR: pp. 102 - 106

**Web Links**

Permutations and combinations activities using TI graphing calculators

<http://education.ti.com/calculators/timath/>

keyword: too many choices and permutations and combinations

The following site provides examples and a series of problem sets related to combinations.

<http://www.bmlc.ca/PureMath30>

Keyword: combinatorics lesson 2

## Probability

### Outcomes

*Students will be expected to*

P5 & P6 Continued...

#### Achievement Indicators:

P5.4 *Solve an equation that involves factorials.*

P6.4 *Solve a contextual problem that involves combinations and probability.*

LR1 **Analyze puzzles and games that involve numerical and logical reasoning, using problem-solving strategies.**

[CN, ME, PS, R]

### Elaborations—Strategies for Learning and Teaching

Students are expected to solve a variety of equations in the form  ${}_n C_r = k$  for  $n$ . Ask students to solve an equation such as  ${}_n C_2 = 15$ . They should be able to easily verify that the value of  $n$  must be greater than or equal to  $r$  in  ${}_n C_r$  having explained the reasoning when working with permutations. If they understand that the notation  ${}_n C_r$  means choosing  $r$  elements from a set of  $n$  elements, it naturally flows that  $r$  cannot be larger than  $n$ .

At this point in the unit, students have solved many problems that involve permutations and combinations independently. They should now progress to a mixture of problems where they must make a decision as to which concept applies. When reading a problem, students should be asking themselves questions such as “Does order matter in this problem?” If yes, then they know to solve using permutations; otherwise they will use combinations. For a permutations problem, the next question could be “Are the objects identical or distinguishable?” For a combination problem, the next logical question to follow would be: “Are there multiple tasks (calculations) required?” If yes, then does the Fundamental Counting Principle apply? By developing a systematic approach, students should gain more confidence when faced with novel problem situations.

This would be a good time to incorporate puzzles and games involving numerical and logical reasoning. Refer back to pp. 38-47 for additional information.



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**General Outcome: Develop critical thinking skills related to uncertainty.**


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**Suggested Assessment Strategies**
*Performance*

- Ask students to create a foldable or flowchart outlining a series of questions or problem-solving strategies for problems involving permutations or combinations.

(P6.4)

*Paper and Pencil*

- Ask students to solve  ${}_{n+1}C_1 = 20$ .

(P5.4)

**Resources/Notes**
**Authorized Resource**
*Principles of Mathematics 12*
**2.6 Combinations**

SB: pp. 111 - 120

TR: pp. 102 - 106

**2.7 Solving Counting Problems**

SB: pp. 121 - 128

TR: pp. 107 - 113

**Note:**

Question 15d on p. 119 of the SB is not an expectation in this course.

**Games and Puzzles**
**Analyzing a Traditional Game**

SB: p.133

TR: pp. 116-118



# Probability

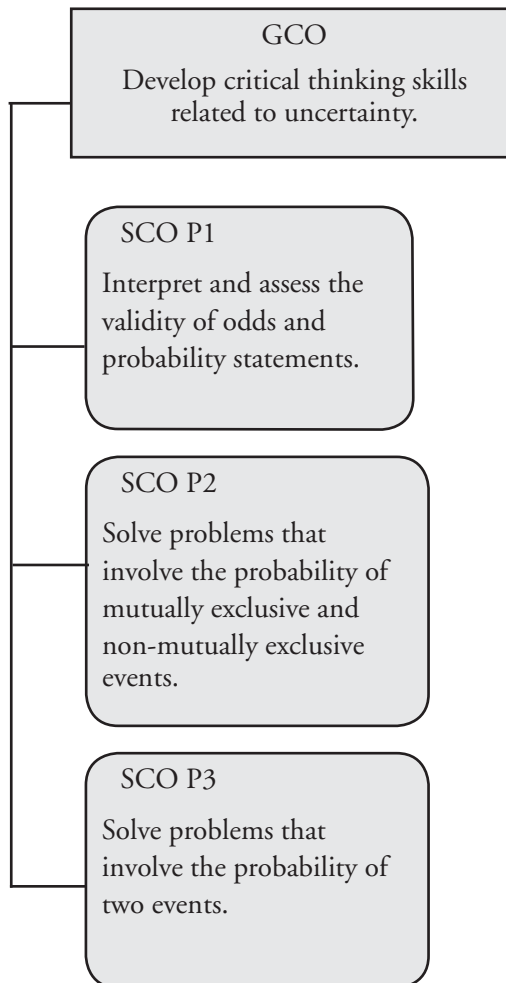
Suggested Time: 13 Hours

# Unit Overview

## Focus and Context

In this unit, students will solve contextual problems involving odds and probability. They will then progress to determining the probability using counting methods. Students will re-visit the Fundamental Counting Principle, permutations and combinations. They will use set theory to develop the formulas for probabilities of events that are mutually exclusive and not mutually exclusive.

## Outcomes Framework



Mathematical Processes

- [C] Communication
- [CN] Connections
- [ME] Mental Mathematics and Estimation
- [PS] Problem Solving
- [R] Reasoning
- [T] Technology
- [V] Visualization

SCO Continuum

Mathematics 1201	Mathematics 2201	Mathematics 3201
Probability		
not addressed	not addressed	<p>P1 Interpret and assess the validity of odds and probability statements. [C, CN, ME]</p> <p>P2 Solve problems that involve the probability of mutually exclusive and non-mutually exclusive events. [CN, PS, R, V]</p> <p>P3 Solve problems that involve the probability of two events. [CN, PS, R]</p>

## Probability

### Outcomes

*Students will be expected to*

**P1 Interpret and assess the validity of odds and probability statements.**

[C, CN, ME]

### Achievement Indicators:

P1.1 *Provide examples of statements of probability and odds found in fields such as media, biology, sports, medicine, sociology and psychology.*

P1.2 *Explain, using examples, the relationship between odds (part-part) and probability (part-whole).*

P1.3 *Determine the probability of, or the odds for and against, an outcome in a situation.*

### Elaborations – Strategies for Learning and Teaching

In Grade 6, students were introduced to theoretical and experimental probabilities (6SP4). In Grade 7, they expressed probabilities as ratios, fractions and percents (7SP4). In Grade 7 and Grade 8, students determined the probability of two independent events (7SP6, 8SP2). In this unit, students will distinguish between probability and odds. They will calculate probability and odds, and use this information to make decisions.

Statements of probability and odds are referenced frequently in various media and texts such as newspapers, magazines, websites and television. Ask students to find statements about probability and odds, and bring them to class. Statements may relate to political polls, games of chance, sports and social statistics. This activity promotes student discussion of the similarities and differences between these two concepts using interesting contexts found in the media.

When there is uncertainty about the occurrence of an event, students can attempt to measure the chances of it happening with probability. Ask students to consider the following questions:

- Will the Ice Caps or the Pirates be more likely to win the game?
- What is the chance of drawing a king from a deck of cards?
- What is the possibility of it raining today?
- What are the chances of getting tails in one toss of the coin?

Remind students the probability of an event is the ratio of favourable outcomes to the total possible outcomes (part:whole).

Although students have been introduced to probability in previous grades, the concept of odds has not been formally introduced. They may need guidance to be able to distinguish between odds for and odds against. Odds of an event show the ratio of favourable outcomes to unfavourable outcomes (part:part). If a student is trying to calculate the odds of getting a heart in a deck of cards, for example, they will have to consider the probability that they will draw a heart from the deck. Since the probability of choosing a heart is  $\frac{1}{4}$ , the odds in favour would be 1:3. Ask students the following:

- How would this ratio change to determine the odds against a student choosing a heart?
- How are the odds in favour related to the odds against?

Students should recognize the formula for the odds against is the reciprocal of finding odds in favour of an event occurring.

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**General Outcome: Develop critical thinking skills related to uncertainty.**


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**Suggested Assessment Strategies**
*Journal*

- Provide students with a newspaper, magazine editorial or an opinion piece. Ask them to identify examples of probability or odds present in the article. Ask them to describe how the author uses probability to support the argument being put forth.

(P1.1)

*Presentation*

- Ask students to create a digital or physical collage of probabilities or odds found in an online news site, a magazine or newspaper.

(P1.1)

- In groups of two, ask students to calculate and explain to the class the odds (or probability) of selecting an 'e' from the word Iceberg. Select similar tasks for the other groups.

(P1.3)

*Paper and Pencil*

- Ask students to identify the following statements as odds or probability:
  - The chances of rolling a 1 on a fair six-sided die is  $\frac{1}{6}$ .
  - The chances of drawing a 4 from a standard 52-card deck is 1:12.

(P1.2)

- Ask students to answer the following questions related to odds and probability:

- The odds of winning a contest are 5:9. What is the probability of winning the contest?
- The probability of you passing the next Math test is 75%. What are the odds of you passing?
- A jar contains three red marbles and some green marbles. The odds are 3:1 that a randomly chosen marble is green. How many green marbles are in the jar?

(P1.3)

**Resources/Notes**
**Authorized Resource**
*Principles of Mathematics 12*
**3.1 Exploring Probability**

Student Book (SB): pp. 140 -141

Teacher Resource (TR): pp. 145-147

**3.2 Probability and Odds**

SB: pp. 142 - 150

TR: pp. 148 - 153

## Probability

### Outcomes

*Students will be expected to*

P1 Continued...

#### Achievement Indicators:

P1.4 *Express odds as a probability and vice versa.*

P1.5 *Solve a contextual problem that involves odds or probability.*

P1.6 *Explain, using examples, how decisions may be based on probability or odds and on subjective judgments.*

### Elaborations—Strategies for Learning and Teaching

Provide students with a variety of examples where they are asked to express odds as a probability and vice versa. Examples can be found or created based on school statistics (e.g., the number of biology students expected this year in Grade 12) or sports (e.g., the chance of a team winning a championship). All odds and probability calculations begin with two of three values: total possibilities, favourable outcomes and non-favourable outcomes. Consider an example such as, the odds in favour of drawing an ace in a standard deck of cards is 4:48 (or 1:12). Ask students the following questions:

- What does this ratio represent?
- What does the sum of the parts of the ratio represent?
- What are the odds against selecting an ace?
- What is the probability of selecting an ace? not selecting an ace?
- How does the probability of selecting an ace and the probability of its complement relate to the odds in favour of selecting an ace?
- Why is it important to know if the ratio represents odds against or odds in favour when using odds to determine the probability?

Students should conclude that if the odds in favour of an event is  $a:b$ , then the total number of possibilities is represented by  $a+b$ . Hence, the probability of an event occurring is  $\frac{a}{a+b}$ . The odds against an event, however, is  $b:a$ . Students have the option to find the probability of an event not happening using the complement or the ratio  $\frac{b}{a+b}$ .

This is a good opportunity for students to conduct in-class surveys and calculate the odds and probability based on the results. A question on the survey could be "How many students use Twitter daily." Other student interests may include music, celebrities, school team results, or sports teams.

Some people use, or choose not to use, probability and odds to make daily decisions. The probability of winning a lottery is extremely low, but millions of people make the subjective judgement that it is still worth the expense. Ask students who has heard the phrase, "Someone has to win". Ask students to debate the merits of this statement.

Encourage students to make decisions based on probability and odds, and to give reasons for their answers.



**General Outcome: Develop critical thinking skills related to uncertainty.**

### Suggested Assessment Strategies

#### *Journal*

- A test is usually conducted to determine blood glucose levels to help assist in diagnosing Type 2 diabetes. There is a 10% chance that a person will have diabetes given a high blood glucose level. 85% of the time individuals with diabetes will register a high blood glucose level, and 99% of the time people without diabetes will show normal blood glucose levels. Ask students to determine, based on the information given, if this an effective test. They should explain their reasoning.

(P1.5)

#### *Paper and Pencil*

- Ask students which of the following "odds for" and "probability" statements are correct. They should explain their reasoning.

	odds for	probability
I	1:3	$\frac{1}{3}$
II	4:5	$\frac{4}{9}$
III	4:6	$\frac{2}{5}$

(P1.4)

### Resources/Notes

#### Authorized Resource

*Principles of Mathematics 12*

#### 3.2 Probability and Odds

SB: pp. 142 - 150

TR: pp. 148 - 153

## Probability

### Outcomes

*Students will be expected to*

**P5 Solve problems that involve permutations.**

[ME, PS, R, T, V]

**P6 Solve problems that involve combinations.**

[ME, PS, R, T, V]

### Achievement Indicators:

*P5.9 Solve a contextual problem that involves probability and permutations.*

*P6.4 Solve a contextual problem that involves probability and combinations.*

### Elaborations – Strategies for Learning and Teaching

Students continue to use counting techniques to solve probability problems. They will determine the probability of an event by dividing the number of ways that the favourable outcome can occur by the total number of possible outcomes.

Students were exposed to the Fundamental Counting Principle, permutations and combinations in the previous unit. They should now use these concepts to solve probability problems. Remind students that permutations are used when the order of the arrangement is important and combinations are used when order is not important. Suppose, for example, a survey was conducted of 500 adults who wore Halloween costumes to a party. Each person was asked how he/she acquired the costume: created it, rented it, bought it, or borrowed it. The results are as follows: 360 adults created their costumes, 60 adults rented their costumes, 60 adults bought their costumes and 20 adults borrowed their costumes. Ask students to determine the probability that the first four people who were polled all created their costumes. Guide students as they solve problem by asking them the following questions:

- What counting strategy will you use? Is order important?
- What are the total number of possible outcomes? What are the number of favourable outcomes?
- What are the number of ways to choose 4 people from the group who created their own costumes?
- How would you use this information to determine the probability?

Students should recognize they will use combinations to find the probability and the total number of possible outcomes can be determined by choosing 4 people from the 500 adults at the party ( ${}_{500}C_4$ ). They will then determine the number of combinations that are possible to choose four adults from the 360 adults who created their own costumes ( ${}_{360}C_4$ ). The probability can then be calculated using the ratio of these two numbers ( $P = \frac{{}_{360}C_4}{{}_{500}C_4}$ ).

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**General Outcome: Develop critical thinking skills related to uncertainty.**


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**Suggested Assessment Strategies**
*Paper and Pencil*

- Ask students to answer the following:
  - (i) A 4-digit PIN number can begin with any digit, except zero, and the remaining digits have no restriction. If repeated digits are allowed, find the probability of the PIN code beginning with a number greater than 7 and ending with a 3.
  - (ii) Mark, Abby and 5 other students are standing in a line. Determine the probability Mark and Abby are standing together. Determine the probability Mark and Abby are not standing together.

P(5.9)

*Interview*

- Two students passed in their response to the following question. There are 7 teachers and 3 administrators at a conference. Find the probability of three different door prizes being awarded to teachers only.

$$\text{Student A} \quad \frac{{}_7C_3 \times {}_3C_0}{{}_{10}C_3} = \frac{35}{120}$$

$$\text{Student B} \quad \frac{{}_{10}C_3 - ({}_7C_2 \times {}_3C_1) - ({}_7C_1 \times {}_3C_2) - ({}_7C_0 \times {}_3C_3)}{{}_{10}C_3} = \frac{35}{120}$$

Ask students the following questions:

- (i) What does student's A equation represent?
- (ii) What does student's B equation represent?
- (iii) Which method would you prefer and why?

P(5.9)

**Resources/Notes**
**Authorized Resource**
*Principles of Mathematics 12*
**3.3 Probabilities Using Counting Methods**

SB: pp. 151 - 162

TR: pp. 154 - 162

## Probability

### Outcomes

*Students will be expected to*

P5, P6 Continued ...

### Achievement Indicators:

P5.9, P6.4 *Continued*

### Elaborations—Strategies for Learning and Teaching

The context of the problem will help students determine the counting technique. Provide students with an example similar to the following:

- If a 4-digit number is generated at random from the digits 2, 3, 5 and 7 (without repetition of the digits), what is the probability that it will be even?

When students encounter a diversity of problems, they should be thinking about the following questions as they work through the problem:

- What counting strategy will I use in this problem?
- What information do I need to determine the probability?
- How will I determine the total number of 4-digit numbers that can be created from these 4 digits?
- What condition must be met in order for the number to be even?
- How will I determine the total number of 4-digit even numbers?

As they solve the problem, encourage students to reflect on the following:

- In order to create an even number, the last digit must be 2. The last digit, therefore, can only be chosen in one way.
- Use the Fundamental Counting Principle to consider the options for the other digits ( $3 \times 2 \times 1$ ).
- In order to determine the probability, consider the total number of ways to create a 4-digit number ( $4!$ ). The probability that the number is even is  $\frac{6}{24} = \frac{1}{4}$ .

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**General Outcome: Develop critical thinking skills related to uncertainty.**

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**Suggested Assessment Strategies**

*Paper and Pencil*

- Ask students to answer the following:
  - (i) A bookcase contains 6 different math books and 12 different biology books. If a student randomly selects two of these books, determine the probability they are both math or both biology books.
  - (ii) A jar contains 5 red, 7 blue and 5 purple candies. If the total number of candies is 20, determine the probability that a handful of four candies contains one of each colour.

P(6.4)

**Resources/Notes**

**Authorized Resource**

*Principles of Mathematics 12*

**3.3 Probabilities Using Counting Methods**

SB: pp. 151 - 162

TR: pp. 154 - 162

## Probability

### Outcomes

*Students will be expected to*

**P2 Solve problems that involve the probability of mutually exclusive and non-mutually exclusive events.**

[CN, PS, R, V]

#### Achievement Indicators:

P2.1 *Classify events as mutually exclusive or non-mutually exclusive, and explain the reasoning.*

P2.2 *Determine if two events are complementary, and explain the reasoning.*

P2.3 *Represent, using set notation or graphic organizers, mutually exclusive (including complementary) and non-mutually exclusive events.*

P2.4 *Solve a contextual problem that involves the probability of mutually exclusive (including complementary) or non-mutually exclusive events.*

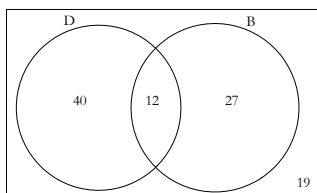
P2.5 *Create and solve a problem that involves mutually exclusive or non-mutually exclusive events.*

### Elaborations – Strategies for Learning and Teaching

The use of set theory is fundamental to the study of probability theory (LR2). Students should continue to use Venn diagrams for illustrating unions and intersections of sets. They will apply the Principle of Inclusion and Exclusion to determine the probability of non-mutually exclusive events and will use the probabilities to make decisions.

Students are familiar with classifying events that are mutually exclusive (disjoint sets) and events that are non-mutually exclusive. They worked with Venn diagrams and represented mutually exclusive sets as sets that did not intersect (i.e.,  $n(A \cap B) = 0$ ). Students are also familiar with an event and its complement. Remind students the outcomes of an event and the outcomes of the complement make up the entire sample space.

Students will now solve probability problems that involve non-mutually exclusive events and mutually exclusive events. It is important for students not only to apply the formula for the probability of two non-mutually events, but to understand how it is developed. Provide students with the following Venn diagram where D represents students on the debate team, B represents students on the basketball team.



Students should recognize that in order to determine the probability of events, they have to think about the following questions:

- Are the two sets intersecting or disjoint?
- How many elements are in each set?
- How many elements are in the universal set  $\mathcal{S}$ ?

Using the Principle of Inclusion and Exclusion, guide students through the following probability formula for non-mutually exclusive events:

$$n(D \cup B) = n(D) + n(B) - n(D \cap B)$$

$$P(D \cup B) = \frac{n(D) + n(B) - n(D \cap B)}{n(\mathcal{S})}$$

$$P(D \cup B) = \frac{n(D)}{n(\mathcal{S})} + \frac{n(B)}{n(\mathcal{S})} - \frac{n(D \cap B)}{n(\mathcal{S})}$$

$$P(D \cup B) = P(D) + P(B) - P(D \cap B)$$

Ask students how they can modify this formula to accommodate events that are mutually exclusive. They should recognize that  $n(D \cap B) = 0$  for disjoint sets, therefore,  $P(D \cup B) = P(D) + P(B)$ .

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**General Outcome: Develop critical thinking skills related to uncertainty.**


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**Suggested Assessment Strategies**
*Paper and Pencil*

- Classify the events in each experiment as either mutually exclusive or non-mutually exclusive:
  - (i) The experiment is rolling a die. The first event is rolling an even number and the second event is rolling a prime number.
  - (ii) The experiment is playing a game of hockey. The first event is that your team scores a goal, and the second event is that your team wins the game.
  - (iii) The experiment is selecting a gift. The first event is that the gift is edible and the second event is that the gift is an iPhone.
- Using the school population as the total, ask students to determine the complement of people doing Mathematics 2201 in your school this year?

(P2.1)

(P2.2)

*Observation*

- Ask students to participate in the *Find Your Partner* activity. Half of the students should be given a card with an event and the other half should be given a card with the complement on it. Students need to move around the classroom to match the event with the correct complement. They should then present their findings to the class.

(P2.2)

**Resources/Notes**
**Authorized Resource**
*Principles of Mathematics 12*
**3.4 Mutually Exclusive Events**

SB: pp. 166 - 181

TR: pp. 163 - 172

**Probability**

**Outcomes**

*Students will be expected to*

P2 Continued...

**Achievement Indicators:**

P2.4, P2.5 *Continued*

**Elaborations – Strategies for Learning and Teaching**

Ask students to create their own examples that involve determining the probability of two mutually exclusive or non-mutually exclusive events. They could exchange problems with a classmate and practice solving various problems.

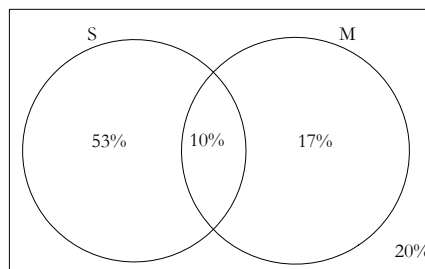
Use numerical examples with students to discuss why the sum of the probability of an event and its complement must equal one. If the probability that a student picks the ace of diamonds, for example, from a standard deck of cards is  $\frac{1}{52}$ , then the probability that he/she will not pick the ace of diamonds is  $\frac{51}{52}$  (i.e.,  $\frac{1}{52} + \frac{51}{52} = 1$ ). Solving the equation  $P(A) + P(A') = 1$  for  $P(A')$ , students should write that  $P(A') = 1 - P(A)$ .

Encourage students to check the reasonableness of their answers. Provide students with the following example:

Class Survey

63% of students play sports  
 27% of the students play a musical instrument  
 20% play neither sports nor a musical instrument

If the events were mutually exclusive, the probabilities of these events and the sum of the complement should total 100%. Students should recognize that  $63\% + 27\% + 20\% = 110\%$ . This implies 10% of the students play sports and play a musical instrument. The events, therefore, are not mutually exclusive. This could be displayed in a Venn diagram as shown below:





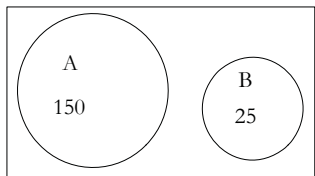
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**General Outcome: Develop critical thinking skills related to uncertainty.**


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**Suggested Assessment Strategies**
*Paper and Pencil*

- Ask students to determine the  $P(A \cup B)$  using the Venn Diagram below. The sample space has 500 outcomes.



(P2.3)

- The probability that Dana will make the hockey team is  $\frac{2}{3}$ . The probability that she will make the swimming team is  $\frac{3}{4}$ . If the probability of Dana making both teams is  $\frac{1}{2}$ , ask students to determine the probability that she will make:
  - at least one of the teams.
  - neither team.

(P2.4)

*Presentation*

- The probability that the Toronto Maple Leafs will win their next game is 0.5. The probability that the Montreal Canadiens will win their next game is 0.7. The probability that they will both win is 0.35. Ask students to determine the probability that one or the other will win (assume they don't play each other). Ask them to create a poster to show their solution that includes a Venn diagram.

(P2.3)

*Observation*

- Ask students to survey the class to determine who brings a pencil to class, a pen to class, or both. Ask them to calculate the probability of bringing a pencil or a pen to class.

(P2.4)

**Resources/Notes**
**Authorized Resource**
*Principles of Mathematics 12*
**3.4 Mutually Exclusive Events**

SB: pp. 166 - 181

TR: pp. 163 - 172

**Web Link**

Develop probability questions using the statistics from the following site:

[http://www.statcan.gc.ca/edu/edu02\\_0000-eng.htm](http://www.statcan.gc.ca/edu/edu02_0000-eng.htm)

## Probability

### Outcomes

*Students will be expected to*

**P3 Solve problems that involve the probability of two events.**

[CN, PS, R]

#### Achievement Indicators:

P3.1 *Compare, using examples, dependent and independent events.*

P3.2 *Determine the probability of two independent events.*

### Elaborations—Strategies for Learning and Teaching

In Grades 7 and 8, students calculated the probability of two independent events (7SP6, 8SP2). They should now distinguish between events that are dependent and independent and use this information when determining the probabilities of two events. To determine whether two events are independent, students will determine whether one event will affect the probable outcome of the other event. If not, the events are independent. If one event does affect the other, then the events are dependent, and students will use conditional probability to calculate the probability of both events occurring.

Students should analyze different events and judge whether the outcome of the first event has an effect on the probability of the second event occurring. Students can use concrete materials, such as a deck of cards, a spinner, or a bag of coins, can be used to show the difference between independent and dependent events. Provide the following scenarios to students to prompt discussion about whether events A and B are independent or dependent.

- Event A: drawing a queen from a standard deck of cards  
Event B: drawing a king from the remaining cards in the same deck
- Event A: rolling a 5 on a die  
Event B: rolling a 3 on the same die

It may be beneficial to discuss the probability of independent events before moving on to problems that involve dependent events. Ask students to use a tree diagram, for example, to model a problem and verify that the probability of two independent events, A and B, is the product of their individual probabilities. Ask students to determine, for example, the probability of rolling a 3 on a die and tossing heads on a coin. Prompt discussion using the following questions:

- Does the outcome of the first event affect the outcome of the second event?
- Using the tree diagram, what is the probability of rolling a three and tossing heads (P(3 and H))?
- What is the probability of rolling a 3 on a die (P(3))? What is the probability of tossing heads on a coin (P(H))?
- What is the value of  $P(3) \times P(H)$ ?
- What did you notice about the value  $P(3) \times P(H)$  and the value from the tree diagram (P(3 and H))?

Discuss with students that when events are independent of each other, the probability of event B does not depend on the probability of event A occurring. In such cases,  $P(A \text{ and } B) = P(A) \times P(B)$ .

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**General Outcome: Develop critical thinking skills related to uncertainty.**

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**Suggested Assessment Strategies***Interview*

- Ask students to classify the following events as either independent or dependent and explain why:
  - (i) The experiment is rolling a die and flipping a coin. The first is rolling a six and the second event is obtaining tails.
  - (ii) The experiment is rolling a pair of dice. The first event is rolling an odd number on one die and the second event is rolling an even number on the other die.
  - (iii) The experiment is dealing 5 cards from a standard deck. The first event is that the first card dealt is a spade, the second event is that the second card is a spade, the third event is that the third card is a spade and so on.
  - (iv) The experiment is to sample two members of a family, a mother and her child. The first event is that the mother has blond hair and the second event is that the child has blond hair.

(P3.1)

**Resources/Notes****Authorized Resource***Principles of Mathematics 12***3.6 Independent Events**

SB: pp. 192 - 201

TR: pp. 179 - 188

## Probability

### Outcomes

*Students will be expected to*

P3 Continued...

#### Achievement Indicators:

P3.3 *Determine the probability of an event, given the occurrence of a previous event.*

P3.4 *Create a contextual problem that involves determining the probability of dependent or independent events.*

### Elaborations – Strategies for Learning and Teaching

Students determined the probability of two independent events by multiplying their individual probabilities. They should now determine the probability of two dependent events in a similar way. Ask students to calculate the probability of drawing a card from a deck given two chances, with or without replacement. Discuss with them that if a card is not replaced, the events are dependent. Students should multiply the probability of A and the probability of B after A occurs. Introduce them to the term conditional probability, the notation  $P(B|A)$  and the formula  $P(A \text{ and } B) = P(A) \times P(B|A)$ . Ask students to rewrite  $P(A \text{ and } B)$  using set notation and manipulate the formula to solve for  $P(B|A)$ :

$$P(A \cap B) = P(A) \times P(B | A)$$

$$\frac{P(A \cap B)}{P(A)} = P(B | A)$$

In order to find  $P(B|A)$ , students should observe, from the formula, that the number of outcomes making up A and the total number of outcomes in the sample space may be affected. Ask students to determine the probability of selecting two diamonds from a standard decks of cards without replacement. They should result in the probability  $\frac{13}{52} \times \frac{12}{51}$  since there is one fewer diamond and one fewer card in the deck.

Depending on the questions posed, encourage students to create a variety of probabilities using techniques learned throughout the unit. Students should be given the opportunity to share their work with others and to question and assess the work of their peers.

Sample questions could include:

- Cards are drawn from a standard deck of 52 cards without replacement. Calculate the probability of obtaining:
  - (i) a king, then another king
  - (ii) a club, then a heart
  - (iii) a black card, then a heart, then a diamond
- Cards are drawn from a standard deck of 52 crads with replacement. Calculate the probability of obtaining:
  - (i) a king, then another king
  - (ii) a club, then a heart
  - (iii) a black card, then a heart, then a diamond

## General Outcome: Develop critical thinking skills related to uncertainty.

### Suggested Assessment Strategies

#### *Paper and Pencil*

- Ask students to answer the following:
  - (i) A jar contains black and white marbles. Two marbles are chosen without replacement. The probability of selecting a black marble and then a white marble is 0.34, and the probability of selecting a black marble on the first draw is 0.47. What is the probability of selecting a white marble on the second draw, given that the first marble drawn was black?
  - (ii) Jane encounters 2 traffic lights on her way to school. There is a 55% chance that she will encounter a red light at the first light, and a 40% chance she will encounter a red light on the second light. The two traffic lights operate on separate timers. Determine the probability that both lights will be red on her way to school.
  - (iii) Results were summarized in a table of individuals "In favour of" or "Against" a decision.

	In Favour (I)	Against (A)	Total
Male (M)	15	45	60
Female (F)	4	36	40
Total	19	81	100

- (i) Ask students to show that the events "male (M)" and "in favour (I)" are dependent. They should compare the results of  $P(M)$  to  $P(M|A)$ .
- (ii) Ask students to determine the probability that the person is "male" and "in favour" of the decision.
 

(P3.2, P3.3)
- (iv) A hockey team has jerseys in three different colours. There are 4 green, 6 white and 5 orange jerseys in the hockey bag. Todd and Blake are given a jersey at random. Students were asked to write an expression representing the probability that both jerseys are the same colour. Which student correctly identified the probability and why?

Tony	$(\frac{2}{4})(\frac{2}{6})(\frac{2}{5})$
Sam	$(\frac{2}{4}) + (\frac{2}{6}) + (\frac{2}{5})$
Lesley	$(\frac{4}{15})(\frac{3}{14}) + (\frac{6}{15})(\frac{5}{14}) + (\frac{5}{15})(\frac{4}{14})$
Dana	$(\frac{4}{15})(\frac{4}{15}) + (\frac{6}{15})(\frac{6}{15}) + (\frac{5}{15})(\frac{5}{15})$

(P3.3, P3.4)

### Resources/Notes

#### Authorized Resource

*Principles of Mathematics 12*

#### 3.5 Conditional Probability

SB: pp. 182 - 191

TR: pp. 173 - 178

#### 3.6 Independent Events

SB: pp. 192 - 201

TR: pp. 179 - 188



# Rational Expressions and Equations

Suggested Time: 12 Hours

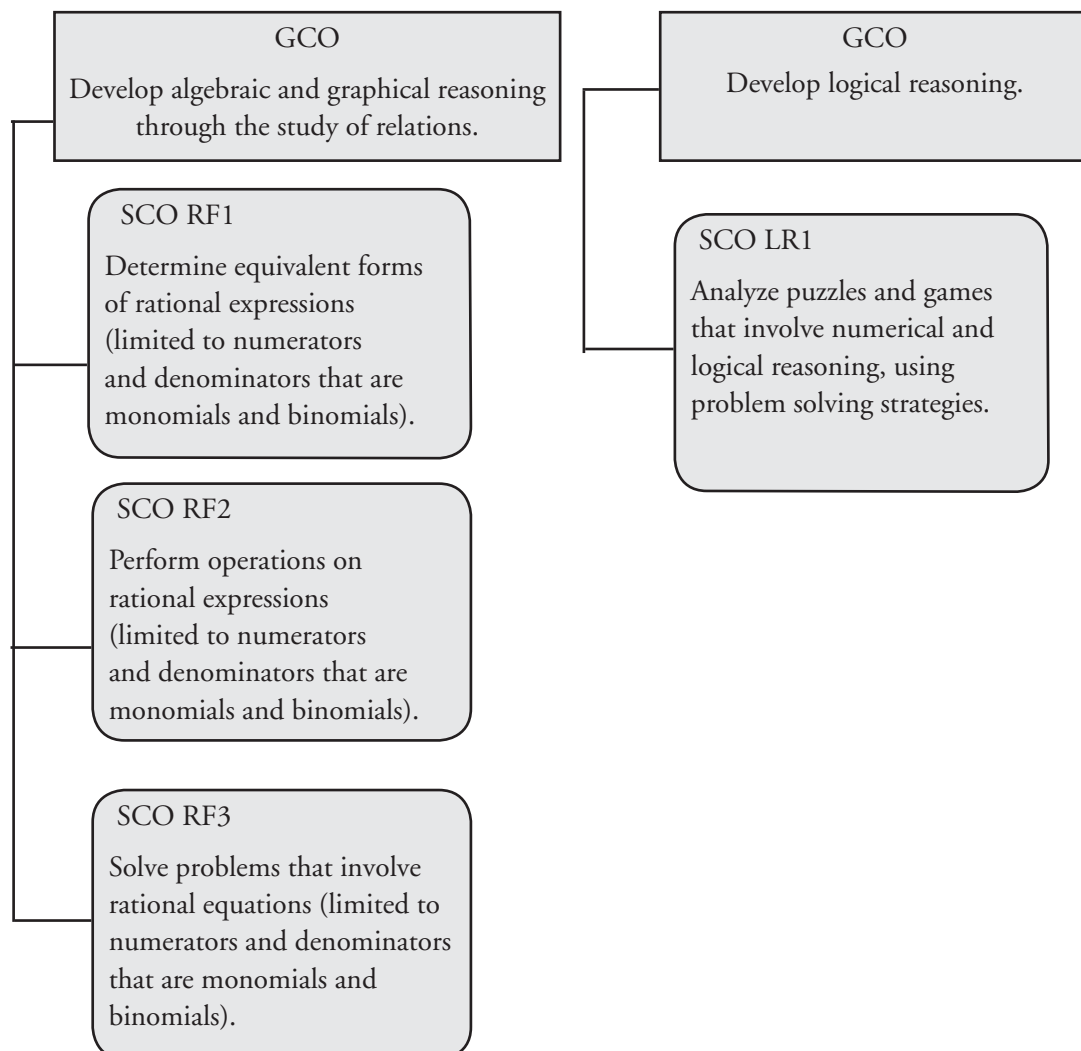
# Unit Overview

## Focus and Context

In this unit, students will simplify rational expressions and determine non-permissible values. They will also add, subtract, multiply and divide rational expressions, limited to numerators and denominators that are monomials and binomials.

Students will solve problems that involve rational equations. They will determine the solution to a rational equation algebraically and identify the non-permissible values.

## Outcomes Framework





# Mathematical Processes

[C] Communication  
 [CN] Connections  
 [ME] Mental Mathematics and Estimation

[PS] Problem Solving  
 [R] Reasoning  
 [T] Technology  
 [V] Visualization

## SCO Continuum

Mathematics 1201	Mathematics 2201	Mathematics 3201
Algebra and Number	Relations and Functions	Relations and Functions
<p>AN5 Demonstrate an understanding of common factors and trinomial factoring, concretely, pictorially and symbolically.                      [C, CN, R, V]</p> <p>AN1 Demonstrate an understanding of factors of whole numbers by determining the:</p> <ul style="list-style-type: none"> <li>• prime factors</li> <li>• greatest common factor</li> <li>• least common multiple</li> <li>• square root</li> <li>• cube root.</li> </ul> <p>[CN, ME, R]</p>	<p>RF2 Solve problems that involve quadratic equations.                      [C, CN, PS, R, T, V]</p>	<p>RF1 Demonstrate equivalent forms of rational expressions (limited to numerators and denominators that are monomials and binomials).                      [C, ME, R]</p> <p>RF2 Perform operations on rational expressions (limited to numerators and denominators that are monomials and binomials).                      [CN, ME, R]</p> <p>RF3 Solve problems that involve rational equations (limited to numerators and denominators that are monomials and binomials).                      [C, CN, PS, R]</p>
	Number and Logic	Logical Reasoning
not addressed	<p>NL4 Solve problems that involve radical equations (limited to square roots or cube roots).                      [C, PS, R]</p> <p>NL2 Analyze puzzles and games that involve spatial reasoning, using problem solving strategies.                      [CN, PS, R, V]</p>	<p>LR1 Analyze puzzles and games that involve numerical and logical reasoning, using problem solving strategies.                      [CN, ME, PS, R]</p>

## Relations and Functions

### Outcomes

*Students will be expected to*

**RF1 Demonstrate equivalent forms of rational expressions (limited to numerators and denominators that are monomials and binomials).**

[C, ME, R]

#### Achievement Indicators:

RF1.1 *Explain why a given value is non-permissible for a given rational expression.*

RF1.2 *Determine the non-permissible values for a rational expression.*

### Elaborations – Strategies for Learning and Teaching

In Grade 9, students solved problems that involved arithmetic operations on rational numbers (9N3). They will now be introduced to rational expressions limited to numerators and denominators that are monomials and binomials. They will simplify them and determine the non-permissible values.

A rational expression is any expression that can be written as the quotient of two polynomials, in the form  $\frac{P(x)}{Q(x)}$  where  $Q(x) \neq 0$ .

To begin work with rational expressions, provide students with several examples of expressions, such as  $\frac{4}{5}$ ,  $\frac{2x}{y}$ ,  $\frac{x^2-4}{x+1}$ ,  $\sqrt{5}$ ,  $2\pi$ ,  $\frac{\sqrt{x}}{2y}$ ,  $\frac{x^2}{4}$  and ask them to identify and explain why an expression is or is not a rational expression. It should be pointed out to students that all rational expressions are algebraic fractions but not all algebraic fractions are rational expressions. In the above list, for example,  $\frac{4}{5}$ ,  $\sqrt{5}$ , and  $2\pi$  are not rational expressions.

Non-permissible values are the values of a variable that make the denominator of a rational expression equal zero. In Grade 7, students were introduced to the concept of why a number cannot be divided by zero (7N1). Students should first find the non-permissible values of a rational expression where the denominator is a first degree polynomial such as  $\frac{x}{x+2}$ , and then progress to second degree polynomials.

A common error occurs when students generalize that the non-permissible value is zero rather than looking at the value(s) of  $x$  that produce a denominator of zero. Encourage them to substitute the non-permissible value(s) for  $x$  back into the denominator to verify the denominator results in zero.

Students should also be exposed to rational expressions where the denominator is a second degree polynomial. Provide students with an expression, such as  $\frac{x-1}{3x^2-12}$ . Ask them to answer the following questions: Using inspection, what value(s) of  $x$  would make the denominator zero? What other strategies can be used to solve the quadratic equation? Are there any rational expressions without non-permissible values?

To solve  $3x^2-12$ , students may remove the greatest common factor (GCF) and apply the zero product property, apply the quadratic formula or the square root property. Ask students which method is more efficient and why. If students use the square root property, they divide the equation  $3x^2 = 12$  by 3 and solve  $x^2 = 4$ . Some may mistakenly write the non-permissible value as  $x = 2$  rather than  $x = \pm 2$ . Another error occurs when students factor  $3x^2 - 12 = 0$  as  $3(x - 2)(x + 2)$  and include 3 as a non-permissible value. Remind them to verify their work by expanding the product of the factors using the distributive property.

## General Outcome: Develop algebraic and graphical reasoning through the study of relations.

### Suggested Assessment Strategies

#### *Paper and Pencil*

- Ask students to write a rational expression for the following non-permissible values of 0, -2 and 3. They should compare their answers with the class.

(RF1.2)

#### *Interview*

- Ask students to explain why  $x = 2$  is a non-permissible value for  $\frac{3x}{x-2}$ .

(RF1.2)

#### *Performance*

- Ask students to work in groups to participate in the *Domino Game*. Provide each group with 10 domino cards. One side of the card should contain a rational expression, while the other side contains non-permissible values for a different rational expression. The task is for students to lay the dominos out such that the non-permissible values on one card will match with the correct rational expression on another. They will eventually form a complete loop with the first card matching with the last card. A sample is shown below:

$\frac{3x}{2x-1} \quad x \neq \frac{1}{3}$	$\frac{x+2}{3x-1} \quad x \neq -\frac{1}{2}$	$\frac{7x}{2x+1} \quad x \neq 2$
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(RF1.1, RF1.2)

### Resources/Notes

#### Authorized Resource

*Principles of Mathematics 12*

#### 4.1: Equivalent Rational Expressions

Student Book (SB): pp. 216 - 224  
Teacher Resource (TR): pp. 211-216

## Relations and Functions

### Outcomes

*Students will be expected to*

RF1 Continued ...

#### Achievement Indicators:

RF1.1, RF1.2 *Continued*

RF1.3 *Compare the strategies for writing equivalent forms of rational expressions to the strategies for writing equivalent forms of rational numbers.*

RF1.4 *Determine a rational expression that is equivalent to a given rational expression by multiplying the numerator and denominator by the same factor (limited to a monomial or a binomial), and state the non-permissible values of the equivalent rational expression.*

### Elaborations—Strategies for Learning and Teaching

It is important for students to differentiate between non-permissible values and inadmissible values. Remind students that inadmissible values were discussed in Mathematics 2201 when they worked with Quadratic Equations (RF2). These are values that do not make sense in a given context. Students should continue to work with inadmissible values for a variable in a rational expression. If a boat traveled 20 km with a speed of  $x$  km/h, for example, the time taken for the trip would be represented by  $\frac{20}{x}$ . If students are asked to determine the slowest speed the boat can travel, they should state that the non-permissible value is 0 but the inadmissible value is  $x < 0$ .

In Grade 7, students developed skills in writing equivalent positive rational numbers (7N7). Students should apply these strategies to rational expressions. This concept is essential when adding and subtracting rational expressions later in this unit. To activate this prior knowledge, ask students to multiply both the numerator and denominator of a rational number such as  $\frac{3}{2}$  by 3, by 5 and by  $-4$ . They should consider whether the value of the original fraction changes.

Students are aware that they can multiply or divide a rational expression by 1 without changing its value. They should now work through the following to discover that a rational expression is not equivalent to another rational expression if their restrictions are different.

- Given the rational expression  $\frac{4}{x}$  where  $x \neq 0$ , ask students to write a rational expression by multiplying both the numerator and denominator by 2, by  $x$ , and by  $x + 1$ . Did any of their expressions produce a new restriction?

Using substitution, ask students to verify if the expressions are equivalent. When the expressions  $\frac{4}{x}$ ,  $x \neq 0$  and  $\frac{4(x+1)}{x(x+1)}$ ,  $x \neq 0, -1$  are compared, they are both undefined at  $x = 0$ . When  $x = -1$ , however, the expression  $\frac{4}{x}$  simplifies to  $-4$  while the expression  $\frac{4(x+1)}{x(x+1)}$  is undefined. Since the expressions are not equal for the same value of  $x$ , the expressions are not equivalent.

When writing an equivalent expression, caution students to use the distributive property appropriately. When simplifying  $\frac{x}{x+4} \times \left(\frac{2}{2}\right)$ , for example, students may incorrectly write  $\frac{2x}{2x+4}$  or  $\frac{2x}{x+8}$ . To avoid this error, encourage students to place brackets around the binomial when multiplying.

**General Outcome: Develop algebraic and graphical reasoning through the study of relations.**

### Suggested Assessment Strategies

#### Journal

- Ask students to respond to the following:

What are the non-permissible values for  $\frac{x+3}{x^2-16}$ ?

The image shows three octagons, each with a stick figure pointing to it. The octagons contain the following text:

- Top-left octagon: "I think the non-permissible value is 4."
- Top-right octagon: "I think the non-permissible values are -4 and 4."
- Bottom-center octagon: "I think the non-permissible values are -4, -3 and 4."

Who is correct? Justify your answer by solving the problem.

(RF1.1, RF1.2)

- Your friend thinks the expressions  $\frac{3}{2x}$  and  $\frac{3(x+1)}{2x(x+1)}$  are equivalent. Ask students to explain why these expressions are not equivalent.

(RF1.3, RF1.4)

#### Paper and Pencil

- Ask students to complete the following table:

Are the expressions equivalent?	Yes	No	Justify your choice
$\frac{x+3}{x-4}$ and $\frac{4x+12}{4x-16}$			
$\frac{5}{x-5}$ and $\frac{5x+25}{x^2-25}$			
$\frac{x+2}{x-3}$ and $\frac{3x+6}{2x-6}$			

(RF1.3, RF1.4)

### Resources/Notes

#### Authorized Resource

*Principles of Mathematics 12*

4.1: Equivalent Rational Expressions

SB: pp. 216 - 224

TR: pp. 211 - 216

## Relations and Functions

### Outcomes

*Students will be expected to*

RF1 Continued ...

#### Achievement Indicators:

RF1.5 *Simplify a rational expression.*

RF1.6 *Explain why the non-permissible values of a given rational expression and its simplified form are the same.*

RF1.7 *Identify and correct errors in a given simplification of a rational expression, and explain the reasoning.*

### Elaborations—Strategies for Learning and Teaching

Simplifying a rational expression to lowest terms mirrors the process of simplifying fractions. In both cases, common factors in the numerator and denominator form a ratio of one and can be simplified.

It is important for students to understand the benefit of simplifying rational expressions, whether it be for evaluating or performing operations. Ask students to evaluate the expression  $\frac{x^2+4x}{x}$  where  $x \neq 0$  at  $x = 2$ . Then evaluate the expression  $x + 4$  where  $x \neq 0$  at  $x = 2$ . Ask them the following questions:

- Why were the results the same?
- What is the benefit of simplifying an expression before substituting values for the variables?
- Why does the simplified expression include a non-permissible value?

Students should be able to explain that one of the benefits of simplifying an expression is to create an equivalent expression that is easier to evaluate. Stress to students that the simplified expression must retain the non-permissible values of the original expression for both to be equivalent.

Along with providing the correct solutions, it is beneficial for students to identify incorrect solutions, including why errors might have occurred and how they can be corrected.

When simplifying rational expressions, students might incorrectly cancel terms rather than factors. They might simplify, for example,  $\frac{x^2+x}{x^2-1}$  as  $\frac{x^2+x}{x^2-1}$  resulting in  $-x$ . To help students see this error, ask them to make a comparison with a numerical rational expression such as  $\frac{8}{12} = \frac{5+3}{5+7}$  and  $\frac{8}{12} = \frac{5+3}{5+7} = \frac{3}{7}$ . Ask them if  $\frac{8}{12}$  is equal to  $\frac{3}{7}$ . Students should realize that cancelling a portion of the factor is incorrect. Another error occurs when students omit a numerator of 1 after the rational expression is simplified. They mistakenly simplify  $\frac{3}{6x}$ , for example, as  $2x$ . Encourage students to check the reasonableness of their answer in this case by rewriting the the expression as  $\frac{3}{6} \cdot \frac{1}{x}$ .

## General Outcome: Develop algebraic and graphical reasoning through the study of relations.

### Suggested Assessment Strategies

#### Performance

- Ask students to participate in the following activities:
  - (i) Divide the class into two groups. One group will be given rational expressions and the other group will be given the associated rational expression in simplest form. Ask students to find a partner who has a rational expression equivalent to theirs.
 

(RF1.5)
  - (ii) Students create a unique Bingo card for Rational Expression Bingo. Distribute a blank Bingo card to each student. Teachers should predetermine various expressions involving rational expressions they would like students to simplify. The expressions should be placed in a bag, with simplified expressions on the board. Ask students to write one of the simplified expressions in each square. The center square should remain a "free" space. The teacher pulls an expression from a bag. Students simplify the expression, find its value on their card and cross it off. The first person with a straight line or four corners wins, or the first person with an X or a T on the Bingo card could win.
 

(RF1.5)

#### Observation

- Set up centres containing examples of incorrect simplified rational expressions and their non-permissible values. Ask students to move around the centres to identify and correct the errors. A sample is shown below:

$$\frac{8x - 12}{6x^2 - 4x}, x \neq 0, \frac{2}{3}$$

$$\frac{4(2x - 3)}{2x(3x - 2)}$$

$$\frac{4}{2x}(1)$$

$$2x, x \neq 0, \frac{2}{3}$$

(RF1.7)

### Resources/Notes

#### Authorized Resource

*Principles of Mathematics 12*

#### 4.2: Simplifying Rational Expressions

SB: pp. 225 - 231

TR: pp. 217 - 222

## Relations and Functions

### Outcomes

*Students will be expected to*

**RF2 Perform operations on rational expressions (limited to numerators and denominators that are monomials and binomials).**

[CN, ME, R]

### Achievement Indicators:

RF2.1 *Compare the strategies for performing a given operation on rational expressions to the strategies for performing the same operation on rational numbers.*

RF2.2 *Determine the non-permissible values when performing operations on rational expressions.*

RF2.3 *Determine, in simplified form, the product or quotient of rational expressions.*

### Elaborations—Strategies for Learning and Teaching

In Grade 9, students solved problems involving operations on rational numbers (9N3). This will now be extended to adding, subtracting, multiplying and dividing rational expressions with numerators and denominators limited to monomials and binomials.

Multiplying and dividing rational expressions is very similar to the process students used to multiply and divide rational numbers. Using examples such as  $\frac{12}{25} \times \frac{10}{21}$  and  $\frac{x^2-9}{x^2-4x} \times \frac{x-4}{x-3}$ , ask students to simplify and find the product for each. They should think about whether the strategy for multiplying rational expressions is the same as the strategy for multiplying rational numbers. Ask them to consider the step at which the non-permissible values are determined.

Ask students to find the product of  $\frac{x^2}{x^2-4} \times \frac{x+2}{x}$ , for example, using two different strategies. Working through this example should help students understand the importance of factoring the numerator and denominator of the rational expression, if possible, before the product is determined. Ask students which strategy is more efficient and why.

Reinforce that multiplication of rational expressions follows the same procedure as multiplying rational numbers, but with the added necessity of determining the non-permissible values for the variables.

Provide students an opportunity to compare the division of rational numbers to division of rational expressions. Students sometimes forget to identify the non-permissible values for the numerator of the divisor in a division statement. Reinforce the importance of this step through the use of examples.



**General Outcome: Develop algebraic and graphical reasoning through the study of relations.**

### Suggested Assessment Strategies

#### *Paper and Pencil*

- Ask students to create an activity sheet where the column on the left contains operations with rational expressions and the column on the right contains the non-permissible values (not in the same order). Students will then exchange their sheets. The task is to match each expression with its correct non-permissible values (the non-permissible values may match more than one expression on the left or may not match any).

(RF2.2)

#### *Journal*

- Sean stated that the permissible values for the quotient and the product of the expressions  $\frac{2x^2+6x}{x^2-16}$  and  $\frac{x+3}{x^2-16}$  are the same. Ask students if they agree or disagree with his statement. They should justify their answer.

(RF2.2, RF2.3)

### Resources/Notes

#### Authorized Resource

*Principles of Mathematics 12*

#### 4.3: Multiplying and Dividing Rational Expressions

SB: pp. 232 - 239

TR: pp. 223 - 228

## Relations and Functions

### Outcomes

*Students will be expected to*

RF2 Continued ...

#### Achievement Indicators:

RF2.4 *Determine, in simplified form, the sum or difference of rational expressions that have the same denominator.*

RF2.5 *Determine, in simplified form, the sum or difference of two rational expressions that have different denominators.*

RF2.2 *Continued*

### Elaborations – Strategies for Learning and Teaching

In Mathematics 1201, students were introduced to the lowest common multiple for a set of numbers (AN1). Provide students with the following table so they can compare finding the lowest common denominator of rational numbers to finding the lowest common denominator of rational expressions.

Rational Number	Situation	Rational Expression
$\frac{3}{7} - \frac{2}{7}$	the denominators are the same	$\frac{x^2}{x+1} - \frac{1}{x+1}$
$\frac{1}{12} + \frac{5}{6}$	one denominator is a multiple of the other	$\frac{3}{x+5} - \frac{1}{4x+20}$
$\frac{2}{3} + \frac{7}{2}$	the denominators have no common factors	$\frac{3}{2x} + \frac{4}{x-1}$
$\frac{5}{14} + \frac{1}{6}$	the denominators have a common factor	$\frac{7}{x^2-9} + \frac{1}{4x+12}$

Ask students to answer the following questions related to the rational expressions:

- How do you find the lowest common denominator? Why is it beneficial to simplify the expression before finding the lowest common denominator?
- What are the non-permissible values?
- Can you list other examples that fit each situation?

A common student error involves adding or subtracting the numerators without first writing the fractions with a common denominator. For example, students mistakenly add  $\frac{x}{5} + \frac{2}{3}$  as  $\frac{x+2}{8}$ .

Remind students to be careful when subtracting rational expressions.

They sometimes forget to distribute the negative sign when there is more than one term in the numerator. For example,  $\frac{3x-2}{(x+2)(x-2)} - \frac{2x-4}{(x+2)(x-2)}$  is often incorrectly written as  $\frac{3x-2-2x-4}{(x+2)(x-2)}$ . Encourage students to use brackets to help them avoid this mistake.

**General Outcome: Develop algebraic and graphical reasoning through the study of relations.**

### Suggested Assessment Strategies

#### *Paper and Pencil*

- Ask students to simplify the rational expressions:

(i)  $\frac{x+7}{2x+14} - \frac{5x}{-3x-21}$

(ii)  $\frac{5x-1}{2x+6} + \frac{16x}{x^2-9}$

(iii)  $\frac{2x+3}{3x-3} - \frac{5x+3}{3x^2-3x}$

(RF2.5)

- Ask students to work in groups to complete the following table. Students should explain the similarities between finding the lowest common denominator (LCM) of two rational numbers versus two rational expressions.

Rational Number	LCM	Rational Expression	LCM	Similarities
$\frac{4}{5} + \frac{3}{5}$		$\frac{6}{2x-1} + \frac{-2}{2x-1}$		
$\frac{1}{5} - \frac{7}{15}$		$\frac{4x}{x-3} - \frac{5}{6x-18}$		
$\frac{7}{12} + \frac{3}{8}$		$\frac{2}{x^2-36} + \frac{4}{3x+18}$		

(RF2.1, RF2.2, RF2.4, RF2.5)

### Resources/Notes

#### Authorized Resource

*Principles of Mathematics 12*

#### 4.4: Adding and Subtracting Rational Expressions

SB: pp. 244 - 252

TR: pp. 229 - 234

## Relations and Functions

### Outcomes

*Students will be expected to*

**RF3 Solve problems that involve rational equations (limited to numerators and denominators that are monomials and binomials).**

[C, CN, PS, R]

### Achievement Indicators:

RF3.1 *Determine the non-permissible values for the variable in a rational equation.*

RF3.2 *Determine the solution to a rational equation algebraically, and explain the strategy used to solve the equation.*

RF3.3 *Explain why a value obtained in solving a rational equation may not be a solution of the equation.*

### Elaborations—Strategies for Learning and Teaching

In Grade 9, students solved linear equations (9PR3). In Mathematics 2201, students solved quadratic equations and identified inadmissible roots (RF2). They will now solve equations containing rational expressions and check if the solutions are permissible values. Roots that are non-permissible are extraneous. It is intended that the rational equations be those that can be simplified to linear and quadratic equations.

Students should use different strategies when solving rational equations. It would be beneficial for teachers to provide examples that are easier to visualize before moving on to more complex equations. Some students may use trial and error to solve an equation such as  $\frac{x}{10} = \frac{2}{5}$ . Others may be able to determine the solution by inspection. Encourage students to discuss their ideas. For example, a student may respond that in order to get the number 10, the number 5 must be doubled. Therefore, 2 is also doubled, resulting in  $x = 4$ . This student response is a great lead in to the strategy of creating an equivalent rational equation with common denominators. Ask students to rewrite the rational equation with a common denominator ( $\frac{x}{10} = \frac{4}{10}$ ), setting the numerators equal, resulting in  $x = 4$ .

Another strategy involves eliminating the denominators. Multiplying both sides of the equation by the lowest common denominator, students should be able to simplify the equation and determine the solution.

Once the strategies have been discussed, students should then be exposed to solving more complex rational equations.

When solving  $\frac{3}{x} + \frac{7}{2x} = \frac{1}{5}$ , for example, students may find a common denominator for the left side of the equation and then proceed to solve  $\frac{13}{2x} = \frac{1}{5}$ . They can also multiply both sides of the equation by the lowest common denominator  $(10x)\frac{3}{x} + (10x)\frac{7}{2x} = (10x)\frac{1}{5}$ . It is important, however, for students to recognize it is more efficient to multiply both sides of the original equation by the lowest common denominator.

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**General Outcome: Develop algebraic and graphical reasoning through the study of relations.**

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**Suggested Assessment Strategies**

*Paper and Pencil*

- Provide each student with a rational equation. Ask them to identify the non-permissible roots and then solve the equation. Ask students not to write their name on the paper since student solutions will be collected and then redistributed randomly around the room. Ask students to verify if the solution is correct. If the solution is incorrect, they will identify the error and write the correct solution.

(RF3.1, RF3.2)

**Resources/Notes**

**Authorized Resource**

*Principles of Mathematics 12*

**4.5: Solving Rational Equations**

SB: pp. 253 - 261

TR: pp. 235 - 241

## Relations and Functions

### Outcomes

*Students will be expected to*

RF3 Continued ...

#### Achievement Indicators:

RF3.1, RF3.2, RF3.3 *Continued*

### Elaborations—Strategies for Learning and Teaching

Students can add or subtract the terms on the left hand side or the right hand side of the equation before they cross multiply. This process, however, may lead to an equation where the degree of the polynomial is greater than what they started with. Guide students as they work through the following example:

$$\frac{2x^2+1}{x+3} = \frac{x}{4} + \frac{5}{x+3}$$

$$\frac{2x^2+1}{x+3} = \frac{x^2+3x+20}{4(x+3)}$$

$$4(2x^2 + 1)(x + 3) = (x^2 + 3x + 20)(x + 3)$$

This example results in a cubic equation. Students are only familiar with and responsible for solving quadratic and linear equations. Therefore, multiplying both sides of the equation by the lowest common denominator is the method students should choose.

Caution students that it is necessary to find the non-permissible roots at the beginning of the solution since some rational equations may lead to extraneous roots. Using the equation  $\frac{2x+3}{x+5} + \frac{1}{2} = \frac{-14}{2(x+5)}$ , ask students to answer the following questions:

- What is the non-permissible root? What does this mean?
- What is the solution to the resulting linear equation?
- Why is it important to check the solution by using the original equation?

Students should recognize that solutions which are non-permissible values are extraneous roots. Therefore, they must be eliminated as a valid solution.

When solving rational equations, the modified equation may result in either a linear or quadratic equation. Students should have a choice whether to use the quadratic formula or their factoring skills to solve the quadratic equation. Remind them to verify their solutions to avoid extraneous roots.

Students should be able to write an equation to represent a problem. Ask students how they would write an equation to find two numbers if the sum of a number and its reciprocal is  $\frac{5}{2}$ . Students may begin using trial and error and discuss possible solutions. They should then proceed to write the rational equation  $x + \frac{1}{x} = \frac{5}{2}$  and find the solution. Encourage students to write their own example and share with the class.

RF3.4 *Solve a contextual problem that involves a rational equation.*

**General Outcome: Develop algebraic and graphical reasoning through the study of relations.**

### Suggested Assessment Strategies

#### *Journal*

- Ask students to reflect on the process of solving a rational equation. They should respond in writing to three reflective prompts providing six responses, as shown below, to describe what they learned.

3 things I understand:

- 1.
- 2.
- 3.

2 things I am still struggling with:

- 1.
- 2.

1 thing that I will work on:

- 1.

(RF3.1, RF3.2, RF3.3)

### Resources/Notes

#### Authorized Resource

*Principles of Mathematics 12*

4.5: Solving Rational Equations

SB: pp. 253 - 261

TR: pp. 235 - 241

## Relations and Functions

### Outcomes

*Students will be expected to*

RF3 Continued ...

#### Achievement Indicators:

RF3.4 *Continued*

### Elaborations – Strategies for Learning and Teaching

Students should solve problems where they need to check the inadmissible roots within the context of the problem. Discuss different scenarios that produce inadmissible roots. A negative numerical value, for example, would not make sense if referring to time, height and length.

Students may have difficulty interpreting the information from the word problem and writing the rational equation. Encourage them to use tables and diagrams to help them break down the information. Consider the following example:

- Sherry mows a lawn in 4 hours. Mary mows the same lawn in 5 hours. How long would it take both of them working together to mow the lawn? Pose the following questions to begin a discussion:
  - (i) How much of the lawn would Sherry mow in 1 hour?
  - (ii) How much of the lawn would Mary mow in 1 hour?
  - (iii) How much of the lawn would both mow together in 1 hour?

Completing a table such as the one below should help students organize their information.

	Time to mow lawn (hours)	Fraction of lawn mowed in 1 hr
Sherry	4	$\frac{1}{4}$
Mary	5	$\frac{1}{5}$
Both	$x$	$\frac{1}{x}$

When solving the equation  $\frac{1}{4} + \frac{1}{5} = \frac{1}{x}$ , encourage students to check that the solutions satisfying the original equation are permissible, and, in the case of a word problem, realistic in the context.

Students should also solve word problems where the simplified rational equation is a quadratic equation. In such cases, there may be an inadmissible value that must be rejected in the context of the problem.

Revisit the puzzles and games, focusing on the strategies students are using. Refer back to pp. 38-47 for additional information.

LR1 Analyze puzzles and games that involve numerical and logical reasoning, using problem-solving strategies.

[CN, ME, PS, R]



**General Outcome: Develop algebraic and graphical reasoning through the study of relations.**

**Suggested Assessment Strategies**

*Performance*

- Create pairs of cards with word problems and matching equations to solve the word problems. Distribute the cards amongst the students and have them find their partner by matching the word problem with the corresponding equation. Once they have found their partner, students should work in pairs to solve the equation and verify their solution.

(RF3.2, RF3.4)

*Paper and Pencil*

- A student was given the following word problem:  
It takes Mike 9 hours longer to construct a fence than it takes Jason. If they work together, they can construct the fence in 20 hours. How long would it take Mike to construct the fence alone?  
The student solved the equation  $\frac{20}{t} + \frac{20}{t+9} = 1$  and stated the solutions to the word problem were 36 and -5. Ask students to verify the solution and state whether the student is correct.

(RF3.2, RF3.3, RF3.4)

**Resources/Notes**

**Authorized Resource**

*Principles of Mathematics 12*

4.5: Solving Rational Equations

SB: pp. 253 - 261

TR: pp. 235 - 241

**Games and Puzzles**

**The Rational Expressions Dice Game**

SB: pp.267

TR: pp. 244-245



# Polynomial Functions

Suggested Time: 11 Hours

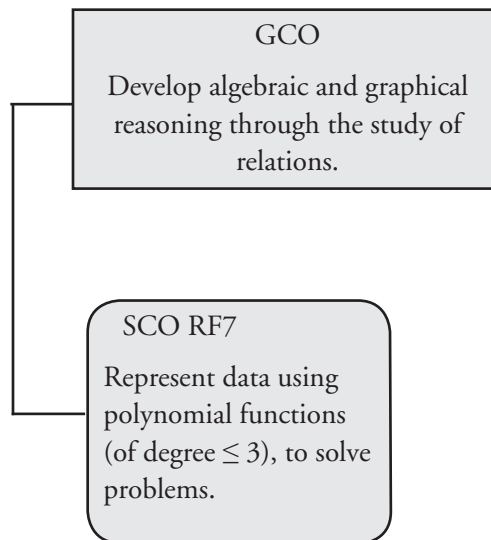
# Unit Overview

## Focus and Context

In this unit, students investigate graphs of polynomial functions (of degree  $\leq 3$ ). They will discover that the graphs of polynomial functions of the same degree have common characteristics. Features such as domain, range, intercepts, turning point, and end behaviour will be used to describe the characteristics of the graphs of polynomial functions.

Using technology, students will create scatter plots for data and use linear, quadratic and cubic regression to model functions.

## Outcomes Framework



## Mathematical Processes

[C] Communication  
 [CN] Connections  
 [ME] Mental Mathematics  
 and Estimation

[PS] Problem Solving  
 [R] Reasoning  
 [T] Technology  
 [V] Visualization

## SCO Continuum

Mathematics 1201	Mathematics 2201	Mathematics 3201
Relations and Functions	Relations and Functions	Relations and Functions
<p>RF5 Determine the characteristics of the graphs of linear relations, including the</p> <ul style="list-style-type: none"> <li>• intercepts</li> <li>• rate of change</li> <li>• domain</li> <li>• range.</li> </ul> <p>[CN, PS, R, V]</p>	<p>RF1 Determine an understanding of the characteristics of quadratic functions, including:</p> <ul style="list-style-type: none"> <li>• vertex</li> <li>• intercepts</li> <li>• domain and range</li> <li>• axis of symmetry.</li> </ul> <p>[CN, PS, T, V]</p>	<p>RF7 Represent data using polynomial functions (of degree <math>\leq 3</math>), to solve problems.</p> <p>[C, CN, PS, T, V]</p>
<p>RF6 Relate linear relations expressed in:</p> <ul style="list-style-type: none"> <li>• slope-intercept form <math>y = mx + b</math></li> <li>• general form <math>Ax + By + C = 0</math></li> <li>• slope-point form <math>y - y_1 = m(x - x_1)</math> to their graphs.</li> </ul> <p>[CN, R, T, V]</p>	<p>RF2 Solve problems that involve quadratic equations.</p> <p>[C, CN, PS, R, T, V]</p>	

## Relations and Functions

### Outcomes

*Students will be expected to*

**RF7 Represent data using polynomial functions (of degree  $\leq 3$ ), to solve problems.**

[C, CN, PS, T, V]

### Achievement Indicator:

RF7.1 *Describe, orally and in written form, the characteristics of a polynomial function by analyzing its graph.*

### Elaborations – Strategies for Learning and Teaching

In Mathematics 1201, students were introduced to the characteristics of linear functions (RF5). In Mathematics 2201, they analyzed quadratic functions to identify the characteristics of the corresponding graph including the vertex, intercepts, domain and range, and the axis of symmetry (RF1). In this unit, students work with polynomial functions of degree 3 or less. They will compare polynomial functions with respect to degree, sign of the leading coefficient (end behaviour) and the constant term ( $y$ -intercept).

Students will determine the linear, quadratic or cubic function that best fits a set of data, and use the function to solve a problem.

Linear and quadratic functions are examples of polynomial functions that students have already studied. They should now extend their study of polynomials to include cubic functions. Students have previously sketched the graphs of polynomial functions of degree 0, 1 and 2.

Function	Degree	Type of Function	Graph
$f(x) = a$	0	constant	horizontal line
$f(x) = ax + b$	1	linear	line with slope $a$
$f(x) = ax^2 + bx + c$	2	quadratic	parabola

Provide students with graphs of various linear, quadratic and cubic functions or students can use graphing technology to sketch the graphs. They will identify patterns in the graphs related to the:

- maximum number of  $x$ -intercepts
- $y$ -intercept
- domain
- range
- end behaviour
- number of turning points.

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**General Outcome: Develop algebraic and graphical reasoning through the study of relations.**

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**Suggested Assessment Strategies**

*Performance*

- This activity will involve a group of two students. Give one student the graph of a polynomial function. Ask him/her to turn to their partner and describe, using the characteristics of the function, the graph they see. The other student will draw the graph based on the description from the student. The students should explain why two graphs could fit the description but still look different from each other.

(RF7.1)

**Resources/Notes**

**Authorized Resource**

*Principles of Mathematics 12*

**5.1 Exploring the Graphs of Polynomial Functions**

Student Book (SB): pp. 274 - 277

Teacher Resource (TR): pp. 266-269

## Relations and Functions

### Outcomes

Students will be expected to

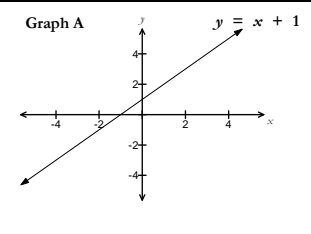
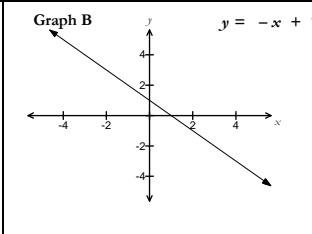
RF7 Continued...

#### Achievement Indicator:

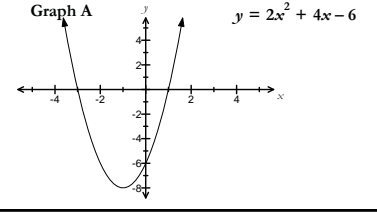
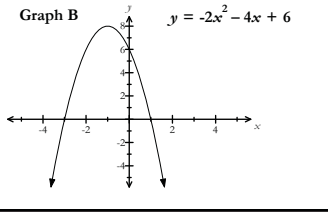
RF7.1 *Continued*

### Elaborations – Strategies for Learning and Teaching

Teachers should begin by explaining the definition of end behaviour of the graph of a function as the behaviour of the  $y$ -values as  $x$  becomes large in the positive or negative direction. The following are two possibilities for oblique linear functions. Ask students to fill in the table and examine the relationship between the slope of a line and the end behaviour.

<p><b>Graph A</b></p> 	<p><b>Graph B</b></p> 
Slope: positive	Slope: negative
End Behaviour: falls to the left and rises to the right (i.e., $y \rightarrow -\infty$ as $x \rightarrow -\infty$ and $y \rightarrow \infty$ as $x \rightarrow \infty$ )	End Behaviour: falls to the right and rises to the left (i.e., $y \rightarrow -\infty$ as $x \rightarrow \infty$ and $y \rightarrow \infty$ as $x \rightarrow -\infty$ )
Line extends from Quadrant III to Quadrant I	Line extends from Quadrant II to Quadrant IV

Similarly, students could use the following graphs to explore the end behaviour of parabolas. Ask students to fill in the table and examine the relationship between the leading coefficient, range, and end behaviour.

<p><b>Graph A</b></p> 	<p><b>Graph B</b></p> 
Leading coefficient is positive	Leading coefficient is negative
Range: $\{y \mid y \geq -8, y \in \mathbb{R}\}$	Range: $\{y \mid y \leq 8, y \in \mathbb{R}\}$
End Behaviour: rises to the left and rises to the right	End Behaviour: falls to the left and falls to the right
Parabola extends from Quadrant II to Quadrant I	Parabola extends from Quadrant III to Quadrant IV

Students should conclude the graph has the same behaviour to the left and right.



**General Outcome: Develop algebraic and graphical reasoning through the study of relations.**

### Suggested Assessment Strategies

#### *Journal*

- Ask students to use graphing technology to determine any similarities and differences between polynomials such as the following:

(i)  $f(x) = 2x + 1$

(ii)  $f(x) = x^2 + 1$

(iii)  $f(x) = x^2 + 4x + 4$

(iv)  $f(x) = (x + 3)^2 - 1$

(v)  $f(x) = 2x^3$

(vi)  $f(x) = x^3 + 2x^2 - x - 2$

They should then graph each of these with a negative leading coefficient.

(RF7.1)

### Resources/Notes

#### Authorized Resource

*Principles of Mathematics 12*

#### 5.1 Exploring the Graphs of Polynomial Functions

SB: pp. 274 - 277

TR: pp. 266-269

#### Web Link

<http://www.padowan.dk/graph>

This site provides a free download of graphing software.

## Relations and Functions

### Outcomes

Students will be expected to

RF7 Continued...

#### Achievement Indicator:

RF7.1 *Continued*

### Elaborations – Strategies for Learning and Teaching

When working with cubics, students should fill in the table and examine the relationship between the leading coefficient, range and end behaviour.

<p><b>Graph A</b></p> <p><math>y = x^3 - 2x^2 - 5x + 6</math></p>	<p><b>Graph B</b></p> <p><math>y = -x^3 + 2x^2 + 5x - 6</math></p>
Leading coefficient is positive	Leading coefficient is negative
Range: $\{y \mid y \in \mathbb{R}\}$	Range: $\{y \mid y \in \mathbb{R}\}$
End Behavior: falls to the left and rises to the right	End Behavior: rises to the left and falls to the right
Cubic extends from Quadrant III to Quadrant I	Cubic extends from Quadrant II to Quadrant IV

Students should notice a pattern as they analyze the end behaviour of cubic functions. That is, the graph has opposite behaviours to the right and left.

For polynomials of degree 3 or less, students should summarize features such as the following:

- the graph of a polynomial function is continuous
- the degree of the polynomial function determines the shape of the graph
- the maximum number of  $x$ -intercepts is equal to the degree of the function
- there is one  $y$ -intercept for all polynomial functions
- the graph of a polynomial function has only smooth turns
- a function of degree  $n$  has a maximum of  $n - 1$  turns
- the end behaviour of a line or curve is the behaviour of the  $y$ -values as  $x$  becomes large in the positive (i.e., as  $x \rightarrow \infty$ ) or negative direction (as  $x \rightarrow -\infty$ ).
- Discuss vertical lines with students. These lines could have no  $y$ -intercept or an infinite number of  $y$ -intercepts. However, they are not functions because they do not satisfy the vertical line test. This means they are not part of the family of polynomial functions.

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**General Outcome: Develop algebraic and graphical reasoning through the study of relations.**

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### Suggested Assessment Strategies

#### *Paper and Pencil*

- Provide students with a variety of polynomial functions (of degree  $\leq 3$ ). Ask them to determine the following characteristics for each polynomial function:
  - (i)  $y$ -intercept
  - (ii) maximum number of  $x$ -intercepts
  - (iii) end behaviour
  - (iv) domain
  - (v) range
  - (vi) number of turning points

Ask students to form three groups based on the shape of their graphs (linear, quadratic or cubic). They should discuss the characteristics of their graphs, the similarities and the differences.

(RF7.1)

### Resources/Notes

#### Authorized Resource

*Principles of Mathematics 12*

#### 5.1 Exploring the Graphs of Polynomial Functions

SB: pp. 274 - 277

TR: pp. 266-269

## Relations and Functions

### Outcomes

*Students will be expected to*

RF7 Continued...

#### Achievement Indicator:

RF7.2 *Describe, orally and in written form, the characteristics of a polynomial function by analyzing its equation.*

### Elaborations—Strategies for Learning and Teaching

In Mathematics 1201, students related linear relations expressed in slope-intercept form, general form and slope-point form to their graphs (RF6). In Mathematics 2201, they examined quadratic functions expressed in standard form  $f(x) = ax^2 + bx + c$ , vertex form  $f(x) = a(x - h)^2 + k$  and factored form  $f(x) = a(x - r)(x - s)$  (RF1). In this unit, students work with cubic equations in the form  $f(x) = ax^3 + bx^2 + cx + d$ .

Students should describe the characteristics of the graph using the degree of the function, the sign of the leading coefficient, and the value of the constant term. Using graphing technology such as winplot or FX Draw, ask students to investigate several equations and their corresponding graphs. It may be beneficial for students to initially graph  $y = x^3$  and  $y = -x^3$ . They should then proceed to other variations such as  $y = 2x^3 + 10x^2 - 2x - 8$  and  $y = -x^3 + x^2 + 5x + 3$ . Use prompts such as the following to promote discussion:

- How does the sign of the leading coefficient in a polynomial function affect the end behaviour of the graph?
- What is the relationship between the constant term in the polynomial function and the  $y$ -intercept of the graph?

Students should observe that the degree of a polynomial function determines the shape of the function. For a cubic function, the end behaviour is opposite on the left and right sides of the graph. If the leading coefficient of the cubic function is positive, then the graph falls to the left and rises to the right. It extends down into Quadrant III and up into Quadrant I (similar to the graph of  $y = x$ ). If the leading coefficient is negative, then the graph rises to the left and falls to the right. It extends up into Quadrant II and down into Quadrant IV (similar to  $y = -x$ ). The constant term is the  $y$ -intercept of the graph.

When presented with a polynomial function, students are only expected to identify the possible number of  $x$ -intercepts and the number of turning points. A cubic polynomial function, for example, with zero turning points will have only one  $x$ -intercept, but a cubic with two turning points could have one, two or three  $x$ -intercepts. Students are not required to factor an equation given in standard form to determine the value of the  $x$ -intercepts or rearrange equations to obtain standard form.

## General Outcome: Develop algebraic and graphical reasoning through the study of relations.

### Suggested Assessment Strategies

#### *Interview*

- Ask students the following questions related to polynomial functions of degree  $\leq 3$ . They should explain their reasoning.
  - (i) Is it possible to have a polynomial function that extends from Quadrant I to IV?
  - (ii) Which type of polynomial function has a maximum or minimum value?
  - (iii) Which type of polynomial function always has a domain and range belonging to the real numbers.
  - (iv) How many turning points are possible if the function is cubic? quadratic? linear?

(RF7.1, RF7.2)

#### *Paper and Pencil*

- For each type of polynomial function ask students to write an equation that has a  $y$ -intercept of  $-3$ .
  - (i) constant
  - (ii) linear
  - (iii) quadratic
  - (iv) cubic
  
- Ask students to write an equation for a polynomial function that satisfies each set of characteristics:
  - (i) extending from Quadrant III to Quadrant IV, one turning point,  $y$ -intercept of  $4$
  - (ii) extending from Quadrant II to Quadrant IV, three  $x$ -intercepts
  - (iii) two turning points,  $y$ -intercept of  $-3$
  - (iv) range of  $y \geq 2$ ,  $y$ -intercept  $2$
  - (v) increasing function, degree  $1$ ,  $y$ -intercept of  $4$

Students should exchange their answers with a partner to recognize their are a variety of possible equations.

(RF7.1, RF7.2)

### Resources/Notes

#### Authorized Resource

*Principles of Mathematics 12*

#### 5.2 Characteristics of the Equations of Polynomial Functions

SB: pp. 278 - 294

TR: pp. 270 - 280

## Relations and Functions

### Outcomes

Students will be expected to

RF7 Continued...

### Achievement Indicator:

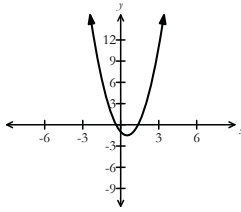
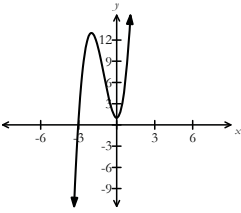
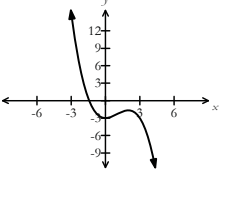
RF7.3 Match equations in a given set to their corresponding graphs.

### Elaborations – Strategies for Learning and Teaching

When analyzing the graph, students will not be expected to describe the  $x$ -intercepts as single, double or triple roots.

Students should be able to match a polynomial function with its corresponding graph. Encourage students to justify their matches in relation to characteristics such as end behaviour, turning points,  $y$ -intercept and number of  $x$ -intercepts.

Provide students with the following table and ask them to provide reasons for their choices to further develop their understanding of the properties of polynomial functions.

Graph	Equation	Reasoning
	$y = 2x^2 - 2x - 1$	<ul style="list-style-type: none"> <li>Leading coefficient is positive, graph rises to the left and rises to the right (extends from Quadrant II to Quadrant I)</li> <li>Constant term is 1, <math>y</math>-intercept is -1</li> <li>degree 2; maximum of two <math>x</math>-intercepts; one turning point</li> </ul>
	$y = 3x^3 + 9x^2 + 1$	<ul style="list-style-type: none"> <li>Leading coefficient is positive, graph falls to the left and rises to the right (extends from Quadrant III to Quadrant I)</li> <li>Constant term is 1, <math>y</math>-intercept is 1</li> <li>degree 3; maximum of three <math>x</math>-intercepts, can have zero or two turning points</li> </ul>
	$y = -\frac{1}{3}x^3 + x^2 - 3$	<ul style="list-style-type: none"> <li>Leading coefficient is negative, graph rises to the left and falls to the right (extends from Quadrant II to Quadrant IV)</li> <li>Constant term is -3, <math>y</math>-intercept is -3</li> <li>degree 3; maximum of three <math>x</math>-intercepts, can have zero or two turning points</li> </ul>

## General Outcome: Develop algebraic and graphical reasoning through the study of relations.

### Suggested Assessment Strategies

#### *Performance*

- For the activity *Match It Up*, present students with a variety of index cards containing separate information about equations, graphs and a written description. Students move around the classroom trying to locate their matching cards.  
(RF7.1, RF7.2, RF7.3)
- Ask students to create a poster or a multimedia presentation summarizing the characteristics of polynomial functions of degree 3 or less.  
(RF7.1, RF7.2, RF7.3)

#### *Paper and Pencil*

- Ask students to sketch two possible graphs of polynomial functions that satisfy each set of characteristics :
  - (i) two turning points, negative leading coefficient, constant term -5
  - (ii) degree two, one turning point which is a minimum, constant term -3  
(RF7.1, RF7.2, RF7.3)

### Resources/Notes

#### *Principles of Mathematics 12*

#### 5.2 Characteristics of the Equations of Polynomial Functions

SB: pp. 278 - 294

TR: pp. 270 - 280

#### Web Link

This site allows students to graph polynomial functions and analyze characteristics such as domain, range, intercepts and end behaviour.

<http://www.webgraphing.com>

## Relations and Functions

### Outcomes

*Students will be expected to*

RF7 Continued...

#### Achievement Indicators:

RF7.4 *Graph data, and determine the polynomial function that best approximates the data.*

RF7.5 *Interpret the graph of a polynomial function that models a situation, and explain the reasoning.*

RF7.6 *Solve, using technology, a contextual problem that involves data that is best represented by graphs of polynomial functions, and explain the reasoning.*

### Elaborations—Strategies for Learning and Teaching

Students should determine a polynomial function (degree  $\leq 3$ ) that best fits a set of data and use the regression function to solve a problem. It should be noted that students are not expected to determine the equation of a regression line or curve without the use of technology. This will be their first exposure to the concept of regression.

Exponential, logarithmic and sinusoidal regression equations will follow in later units.

Provide students an opportunity to become comfortable with using technology to find the regression equations. Graphing calculators, mobile devices, smartphone and tablet applications, computer programs or online tools, such as Geogebra, can be used to calculate and graph regression lines/curves.

Students should work with linear regression before moving on to a quadratic and a cubic regression. Real-life data is used to produce a table and create a scatter plot for the data. Ensure students distinguish between independent and dependent variables. The scatter plot provides them with a visual representation of the data. Review with students a linear relation that can be expressed in slope-intercept form. They should identify the characteristics of the line of best fit and write the equation in  $y = mx + b$  form. Furthermore, students should be able to determine what the slope and  $y$ -intercept of a best fit line represents for a contextual problem.

Students should then be exposed to data where they determine the quadratic or cubic regression equation. Linear regression can be revisited when the different forms of regression are compared. Encourage students to discuss the characteristics of the graph. If the curve appears parabolic and opens downward, for example, they should realize the leading coefficient of its equation should be negative.

It is important for students to observe that the polynomial regression results in an equation of a line (or curve) that balances the points on both sides of the line (or curve). When working with examples, students should be given the type of regression to apply (linear, quadratic, or cubic). Determining the most appropriate type of regression is not an outcome of this course.

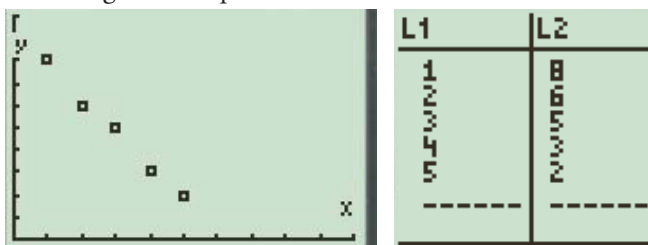


**General Outcome: Develop algebraic and graphical reasoning through the study of relations.**

### Suggested Assessment Strategies

#### Performance

- The following scatter plot shows the change in temperature from 1pm to 5pm in Port Hope Simpson. Ask students to draw lines of best fit for the scatterplot below. Students can share their results with each other and discuss any similarities and differences. Graphing technology should then be used to determine the actual linear regression equation.



## Relations and Functions

### Outcomes

*Students will be expected to*

RF7 Continued...

#### Achievement Indicators:

RF7.4, RF7.5, RF7.6 *Continued*

### Elaborations—Strategies for Learning and Teaching

Provide students with a variety of problems that involves data that is best represented by graphs of polynomial functions.

- The average gasoline price in Canada from 2005 to 2009 is shown. Using a cubic regression, ask students to predict the price of gasoline in the year 2020.

Number of years since 2005	Price in cents/litre
0	92.3
1	87.7
2	101.8
3	114.1
4	94.5

Students can interpolate or extrapolate values by tracing along the line/curve of best fit or by substituting values into the equation of the regression function. Remind students that the regression equation is a model that best suits the data as a whole, rather than at any one point on the scatter plot. Hence, the predicted values calculated may not match experimental data.

Whenever a regression model is used to fit a group of data, the range of the data should be carefully observed. Attempting to use a regression equation to predict values outside of this range (extrapolation) is often inappropriate. Present the following scenarios to students to give them a picture of how extrapolation may yield unrealistic answers.

- A linear model relates weight gain to the age for young children. Applying such a model to adults, or even teenagers, would not be appropriate, since the relationship between age and weight gain is not consistent for all age groups.
- A cubic model relates the number of wins for a hockey team over time (in years). If students extrapolate too far in the future, their model could yield more wins than is possible in a year.

Students will only be expected to estimate, outside the given data, examples that are mathematically appropriate.

External factors that could affect predicted outcomes should also be analyzed. Suppose, for example, data was collected related to the number of shoppers at a local store in a year. If the data follows a cubic trend, the cubic regression could help students make predictions about the number of shoppers at the store on a given day. The model, however, does not predict whether the store decides to have a special sale on that day, or if there could have been increased competition, poor weather conditions, or any other number of factors.

## General Outcome: Develop algebraic and graphical reasoning through the study of relations.

### Suggested Assessment Strategies

#### *Paper and Pencil*

- Ask students to respond to the following email that Mary sent to her teacher.

Hey! I have been working on a formula for the results of my physics project on the amount of weight that can be supported by the thickness of an elastic band. Here is the data I have collected:

I did a quadratic regression and it looks like a pretty good fit.  
The formula I determined was  $W = 0.17t^2 - 0.81t + 5.89$   
( $w$  represents the weight supported (grams) and  $t$  represents the thickness of the band (centimetres))

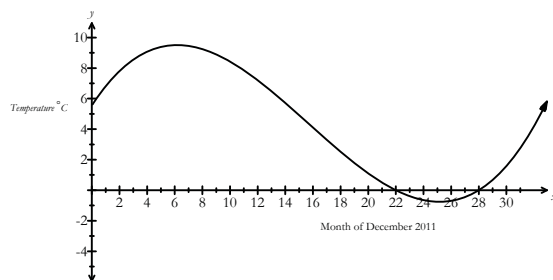
Can you check if this is a good approximation or would a cubic regression be more accurate

Thx!

Thickness of Elastic	Weight
1	1
5	9
10	20
15	34
20	54
25	87
30	141

(RF7.4, RF7.5, RF7.6)

- Ask students to research and collect their own data on a subject of interest. They should create a scatter plot and describe a possible trend in the data. Ask them to determine an appropriate model using regression, pose a problem and then use their model to solve the problem. They should present their findings to the class.  
(RF7.4, RF7.5, RF7.6)
- The following graph was used to model the changes in temperature last December. Ask students to answer the following questions:



- What would your prediction be for the temperature on Jan 1<sup>st</sup>, 2012?
- What was the coldest day of the month?
- What might the temperature have been on Nov. 27<sup>th</sup>, 2011?
- Would you use this model to predict the temperature on Jan. 15<sup>th</sup>, 2012? Explain your reasoning. (RF7.5)

### Resources/Notes

#### Authorized Resource

#### *Principles of Mathematics 12*

#### 5.3 Modelling Data with a Line of Best Fit

SB: pp: 295-306

TR: pp: 281-286

#### 5.4 Modelling Data with a Curve of Best Fit

SB: pp: 307-316

TR: pp: 287-294

#### Web Link

The gasoline prices can be found at:

<http://www.neb-one.gc.ca/clf-nsi/rnrgymfntn/nrgyrprt//gslnprcngnrgfct2010/gslnprcngnrgfct2010-eng.html>



# Exponential Functions

Suggested Time: 13 Hours

# Unit Overview

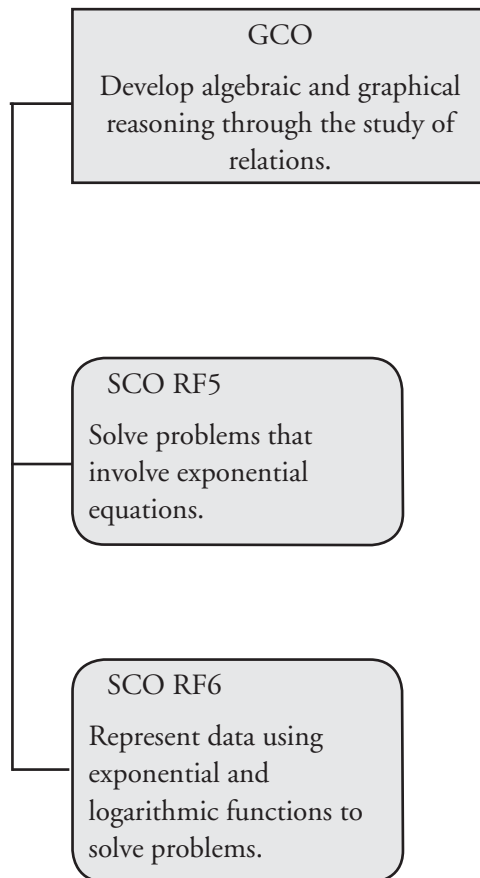
## Focus and Context

Students will describe the characteristics of exponential functions by analyzing their graphs and equations. They will describe aspects such as domain and range, whether it is increasing or decreasing, and the intercepts.

Students will solve exponential equations algebraically using the same base value. In the next unit on logarithms, they will solve exponential equations where the bases are not powers of one another.

Students will solve problems using an exponential regression model. They will construct a scatter plot to display the data and determine the equation of the exponential regression function.

## Outcomes Framework



## Mathematical Processes

- |  |                      |
|--|----------------------|
| [C] Communication                      | [PS] Problem Solving |
| [CN] Connections                       | [R] Reasoning        |
| [ME] Mental Mathematics and Estimation | [T] Technology       |
|  | [V] Visualization    |

## SCO Continuum

Mathematics 1201	Mathematics 2201	Mathematics 3201
Relations and Functions	Relations and Functions	Relations and Functions
RF6 Relate linear relations expressed in: <ul style="list-style-type: none"> <li>slope-intercept form <math>y = mx + b</math></li> <li>general form <math>Ax + By + C = 0</math></li> <li>slope-point form <math>y - y_1 = m(x - x_1)</math> to their graphs.</li> </ul> [CN, R, T, V]	RF2 Solve problems that involve quadratic equations. [C, CN, PS, R, T, V]	RF5 Solve problems that involve exponential equations. [C, CN, PS, R, T]  RF6 Represent data using exponential and logarithmic functions to solve problems. [C, CN, PS, R, T]
Algebra and Number		
AN3 Demonstrate an understanding of powers with integral and rational exponents. [C, CN, PS, R]	not addressed	not addressed

## Relations and Functions

### Outcomes

Students will be expected to

**RF6 Represent data using exponential and logarithmic functions to solve problems.**

[C, CN, PS, R, T]

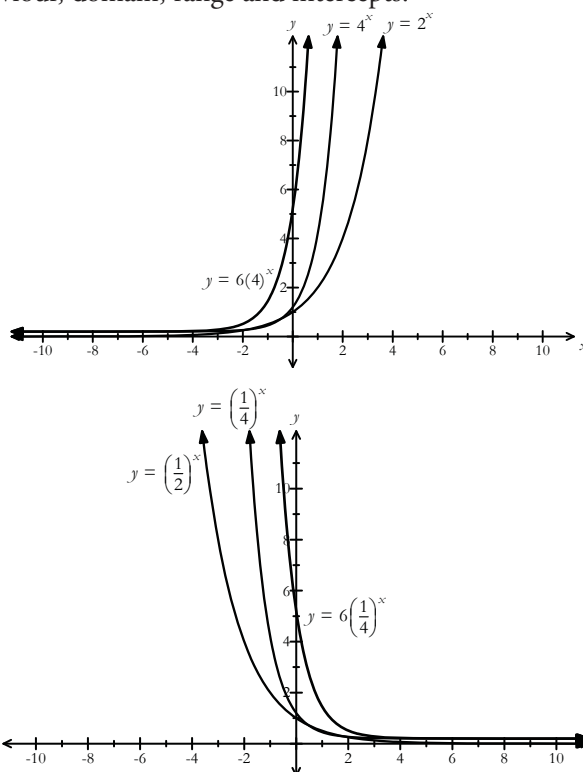
### Achievement Indicator:

RF6.1 Describe, orally and in written form, the characteristics of an exponential function by analyzing its graph.

### Elaborations—Strategies for Learning and Teaching

In Mathematics 1201, students worked with linear relations expressed in slope-intercept form, general form and slope-point form (RF6). In Mathematics 2201, they solved problems that involved quadratic equations (RF2). Students will now be exposed to exponential equations of the form  $y = a(b)^x$ , where  $b > 1$  or  $0 < b < 1$  and  $a > 0$ . Work with logarithmic functions will be completed in the next unit.

Using graphing technology or a table of values, students should investigate the characteristics of an exponential function of the form  $y = a(b)^x$  where  $b > 0$ ,  $b \neq 1$ , and  $a > 0$ . Review terms such as end behaviour, domain, range and intercepts.



Using the graphs as a visual representation, students should note the following characteristics of  $y = a(b)^x$  where  $b > 0$ ,  $b \neq 1$ , and  $a > 0$ :

- no  $x$ -intercepts; one  $y$ -intercept
- exponential functions have a restricted range bounded by the  $x$ -axis but the domain consists of the real numbers
- can be increasing or decreasing
- some exponential functions increase/decrease at a faster rate than others

Students are only exposed to examples where the asymptote is the  $x$ -axis. Asymptotic behaviour is often discussed in courses such as Biology. This would be a good opportunity to discuss the concept of asymptote and its connection to the range of an exponential function.



**General Outcome: Develop algebraic and graphical reasoning through the study of relations.**

### Suggested Assessment Strategies

#### Observation

- Ask students to graph the following functions using graphing technology. They should discuss and record the following information.

	$y = 2^x$	$y = 3^x$	$y = 5(3)^x$
$y$ -intercept			
number of $x$ -intercepts			
end behaviour			
domain			
range			

	$y = (\frac{1}{2})^x$	$y = (\frac{1}{4})^x$	$y = 3(\frac{1}{4})^x$
$y$ -intercept			
number of $x$ -intercepts			
end behaviour			
domain			
range			

(RF6.1)

### Resources/Notes

#### Authorized Resource

*Principles of Mathematics 12*

#### 6.1 Exploring the Characteristics of Exponential Functions

Student Book (SB): pp: 334 - 337  
 Teacher Resource (TR): pp. 326-329

## Relations and Functions

### Outcomes

*Students will be expected to*

RF6 Continued...

#### Achievement Indicators:

RF6.2 *Describe, orally and in written form, the characteristics of an exponential function by analyzing its equation.*

### Elaborations—Strategies for Learning and Teaching

Students should have an opportunity to investigate the equation of an exponential function. They are aware that a polynomial of degree 1 is linear, degree 2 is quadratic and degree 3 is cubic. They should now be exposed to an equation where the power is a variable.

Students should investigate and describe how the parameters  $a$  and  $b$  in the exponential equation  $y = a(b)^x$ , where  $b > 0$ ,  $b \neq 1$ , and  $a > 0$ , affect the graph of the function. They can use graphing technology to create the graph but it is also important to look for patterns in the table of values of the exponential function. Ensure students recognize that when the  $x$ -values increase by 1, the  $y$ -values will increase or decrease by the same multiple. The parameter associated with this multiple is  $b$ .

Ask students to compare, for example,  $y = 2^x$  and  $y = 5(2)^x$ . Similarly, students can compare  $y = (\frac{1}{2})^x$  and  $y = 3(\frac{1}{2})^x$ . Use the following prompts to promote student discussion:

- What is the relationship between the  $y$ -intercept and the parameter  $a$ ?
- How can the  $y$ -intercept be determined algebraically using the equation?
- What happens to the exponential function if  $b > 1$ ? What happens to the  $y$ -values as you move from left to right on the  $x$ -axis?
- What happens to the exponential function if  $0 < b < 1$ ? What happens to the  $y$ -values as you move from left to right on the  $x$ -axis?
- Why does the graph have no  $x$ -intercepts?
- Is the domain or range affected by changing the parameters  $a$  or  $b$ ?

This activity allows students to recognize that the  $y$ -intercept is the  $a$  value and all exponential functions of this form have the same end behaviour, domain and range. The exponential function is increasing if  $a > 0$  and  $b > 1$  and the exponential function is decreasing if  $a > 0$  and  $0 < b < 1$ . Reinforce the value of  $b^0$  and how this relates to the  $y$ -intercept for exponential functions of the form  $y = b^x$  and  $y = a(b)^x$ .

This would be a good opportunity to investigate the graph of  $y = b^x$  where  $b = 1$  and where  $b < 0$ . They should observe the following:

- When  $b = 1$ , a horizontal line is produced.
- When  $b$  is negative, if integer values of  $x$  are chosen the  $y$ -values "oscillate between positive and negative values; for rational values of  $x$ , non-real values of  $y$  may be obtained.

## General Outcome: Develop algebraic and graphical reasoning through the study of relations.

### Suggested Assessment Strategies

#### Interview

- Ask students to describe how  $y = x^2$  is different from  $y = 2^x$ . As teachers listen to responses they should be looking for the descriptions related to domain and range and the shape of the curve.
  - Ask students to respond to the following for  $y = a(b)^x$  where  $b > 0$ ,  $b \neq 1$ , and  $a > 0$ :
    - Why does the function tend to zero as  $x$  gets very large when  $0 < b < 1$ ?
    - Why does the function go to  $\infty$  as  $x$  goes to  $\infty$  when  $b > 1$ ?
- (RF6.2)

#### Performance

- Ask students to participate in the *Find Your Partner* activity. Create pairs of cards with exponential equations and matching graphs. Distribute the cards amongst the students and have them find their partner by matching the equation to its corresponding graph. Once they have found their partner, students should explain why they chose each other.
- (RF6.1, RF6.2)

#### Paper and Pencil

- Ask students to complete the following table using  $y = 8\left(\frac{2}{3}\right)^x$ :

$y = 8\left(\frac{2}{3}\right)^x$	True	False	Why I think so
(i) the $y$ -intercept is 1			
(ii) the graph has one $x$ -intercept			
(iii) the range is $\{y \mid y > 0, y \in \mathbb{R}\}$			
(iv) the domain is $\{x \mid x > 8, x \in \mathbb{R}\}$			
(v) this is a decreasing exponential function			

(RF6.1, RF6.2)

- Using a table of values, ask students to graph  $y = (-2)^x$  and  $y = (1)^x$ . They should explain why the graphs do not represent exponential functions.

(RF6.1, RF6.2)

### Resources/Notes

#### Authorized Resource

*Principles of Mathematics 12*

#### 6.2 Relating the Characteristics of an Exponential Function to Its Equation

SB: pp. 338 - 351

TR: pp. 330 - 335

## Relations and Functions

### Outcomes

*Students will be expected to*

**RF6 Continued...**

**Achievement Indicator:**

RF6.3 Match exponential equations in a given set to their corresponding graphs.

**RF5 Solve problems that involve exponential equations.**

[C, CN, PS, R, T]

**Achievement Indicator:**

RF5.1 Determine the solution of an exponential in which the bases are powers of one another; e.g.,  $2^{x-1} = 4^{x-2}$ .

### Elaborations – Strategies for Learning and Teaching

Students understanding of how the parameters  $a$  and  $b$  can be used to predict characteristics of an exponential function can be consolidated when they match an exponential equation to its corresponding graph. When trying to identify the graph for  $y = 25\left(\frac{1}{5}\right)^x$ , for example, students should be looking for a decreasing exponential function where the  $y$ -intercept is 25.

In Grade 9, students developed and worked with powers having integral bases (excluding base 0) and whole number exponents (9N1, 9N2). In Mathematics 1201, students applied exponent laws to expressions having rational and integral bases, as well as integral and rational exponents. They also expressed powers with rational exponents as radicals and vice versa (AN3). In this unit, students will solve exponential equations.

In Mathematics 2201, students solved quadratic equations (RF2). They should be familiar with solving an equation such as  $x^2 = 64$ . Initiate a discussion as to what would happen if the variable was the exponent,  $2^x = 64$ . Use the following comparison:

	Definition of Exponent	Value of $x$
$x^2 = 64$	$x \times x = 64$	$x = \pm 8$
$2^x = 64$	$2 \times 2 \times 2 \times 2 \times 2 = 64$	$x = 6$

The goal is for students to discover that polynomial functions multiply the variable by itself a fixed number of times while exponential functions vary the number of times a constant is multiplied by itself.

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**General Outcome: Develop algebraic and graphical reasoning through the study of relations.**

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**Suggested Assessment Strategies**

*Performance*

- Pairs of students can play a variation of the card game *Snap*. Create a set of cards containing pairs that display the values of exponential expressions or equations (i.e., one card could have  $2^3$  and the matching card would have 8). Choose a dealer randomly. The dealer shuffles the cards and deals them to the players in the group. Each player places his cards in a pile in front of him face down. Each player turns over the top card from his face-down pile. If the cards on the top of each pile match, the first player to say “Snap!” wins all of the flipped cards. If the player calls “Snap!” when the cards on top of each pile do not match, the opposing player wins all the flipped cards. The first player to run out of cards loses the round. Shuffle the cards and play again.

(RF5.1)

**Resources/Notes**

**Authorized Resource**

*Principles of Mathematics 12*

**6.3 Solving Exponential Equations**

SB: pp. 352 - 365

TR: pp. 336 - 342

## Relations and Functions

### Outcomes

*Students will be expected to*

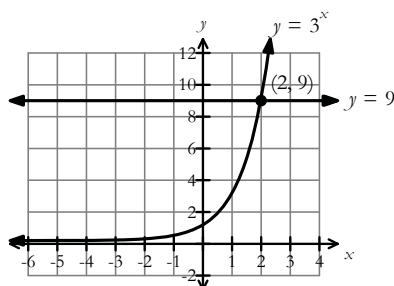
RF5 Continued...

### Achievement Indicator:

RF5.1 *Continued*

### Elaborations – Strategies for Learning and Teaching

Before students solve exponential equations where the bases are powers of one another, they should first be exposed to equations where the bases of both terms are equal. Ask students to solve, for example,  $3^x = 3^2$ . Students may initially write this as  $3^x = 9$  and use inspection to write their solution. It may be beneficial for students to compare the exponential equation,  $y = 3^x$ , to its graph to develop an understanding of the points that satisfy the equation.



Students should observe that the solution to  $3^x = 3^2$  is  $x = 2$ . They sometimes assume they are canceling the bases to reach this solution. It is important, however, to communicate to students they are equating the exponents.

Students should then progress to solving an exponential equation in which the bases are powers of one another. Exponential equations that cannot be solved by rewriting both sides as powers with the same base will be solved algebraically when logarithms are introduced in the next unit.

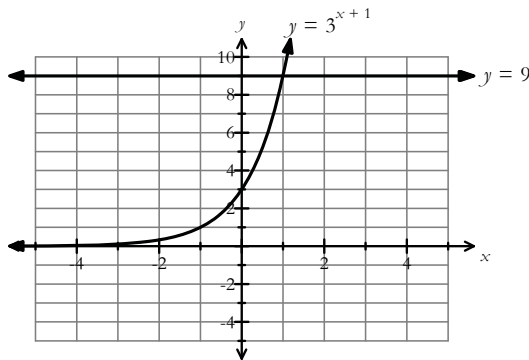
Expose them to the visual representation of an equation such as  $2^{x-1} = 8$  before introducing the algebraic method. When using graphing technology to solve this exponential equation, ask students to write each side of the equation as a graphing function ( $y = 2^{x-1}$  and  $y = 8$ ). Graphing both of these functions on the same axes, they can determine the point of intersection. Ask students which of the values of the shared point represents the solution to the equation. The visual will enhance student understanding of the solution to exponential equations. However, the focus should be on algebraically solving exponential equations.

**General Outcome: Develop algebraic and graphical reasoning through the study of relations.**

### Suggested Assessment Strategies

#### *Pencil and Paper*

- Ask students to use the graph to determine the solution for  $3^{x+1} = 9$ .



They should verify their solution by algebraically determining the solution.

(RF5.1)

#### *Performance*

- Ask students to participate in *Committ and Toss*. Provide students with a selected response problem as shown below. They anonymously commit to an answer and provide a justification for the answer they selected. Students crumple their solutions into a ball and toss the papers into a basket. Once all papers are in the basket, ask students to reach in and take one out. Ask them to then move to the corner of the room designated to match their selected response. In their respective corners, they should discuss the similarities and differences in the explanations provided and report back to class.

Simplify the following:  $5^{2x-2} = \left(\frac{1}{25}\right)^{x-1}$ .

(A)  $5^{2x-2} = 5^{3x-3}$

(B)  $5^{2x-2} = 5^{-3x+1}$

(C)  $5^{2x-2} = 5^{-3x-3}$

(D)  $5^{2x-2} = 5^{-3x+3}$

(RF5.1)

### Resources/Notes

#### Authorized Resource

*Principles of Mathematics 12*

#### 6.3 Solving Exponential Equations

SB: pp. 352 - 365

TR: pp. 336 - 342

**Relations and Functions**

**Outcomes**

*Students will be expected to*

RF5 Continued...

**Achievement Indicator:**

RF5.1 *Continued*

**Elaborations – Strategies for Learning and Teaching**

In order to solve some exponential equations algebraically, students may need to rewrite both sides of the equation as a power of the same base. For example, when students solve,  $2^{x-1} = 8$ , they can rewrite this as  $2^{x-1} = 2^3$  and then proceed to equate the exponents. Remind students they can break down a numerical value as an exponent using personal techniques such as factor trees, prime factorization and/or mental math.

When solving exponentials similar to  $2^{x-1} = 8$ , a common error is to divide by the base instead of using common bases. For example, students may incorrectly solve  $2^{x-1} = 8$  by dividing both sides of the equation by 2 to result in  $x-1 = 4$ . Encourage students to check their answers by substituting the answer back into the equation showing the left side of the equation is equal to the right side of the equation.

Revisit with students the concept of converting from radical form to exponential form. Students should be exposed to examples such as  $\sqrt{8} = 2^{3 \times -4}$ , where they write  $\sqrt{8}$  as  $8^{\frac{1}{2}}$ . They should then proceed to express both sides as powers with the same base. Students should also work with exponentials containing a negative exponent. When solving  $\frac{1}{9} = 3^{x+4}$ , for example, they can write  $\frac{1}{9}$  as  $3^{-2}$  and then solve  $-2 = x + 4$ .

Students should consider what might happen if the power is multiplied by a constant such as  $4(2)^{x-1} = 8$ . Students sometimes incorrectly write this equation as  $8^{x-1} = 8$  instead of using their exponent rules.

When the exponent is a binomial, students sometimes forget to apply the distributive property. When solving the equation  $3 = 81^{x-1}$ , for example, students incorrectly write this as  $3 = 3^{4x-1}$ . Ensure students place brackets around the exponent and then use the distributive property.

Students should develop estimation skills for the solutions of exponentials with variable exponents so that later, when solving equations using logarithms, they will be better able to determine the reasonableness of solutions. Suggested strategies include systematic trial and graphing technology.

• Systematic Trial

Students could solve  $2^x = 10$ , correct to two decimal places, using the process outlined here:

Since 10 is closer to  $2^3 = 8$  than  $2^4 = 16$ , they might begin with  $x \doteq 3.3$ .

Test value for $x$	Power	Approximate value
3.3	$2^{3.3}$	9.849
3.4	$2^{3.4}$	10.556

RF5.2 *Determine the solution of an exponential in which the bases are not powers of one another; e.g.,  $2^{x-1} = 3^{x+1}$ .*



**General Outcome: Develop algebraic and graphical reasoning through the study of relations.**

### Suggested Assessment Strategies

#### Observation

- Set up centers containing examples of incorrect solutions to exponential equations. Ask students to move around the centers to identify and correct the errors. A sample is shown below:

$$\sqrt{5} = 25^{3x+4}$$

$$5^{\frac{1}{2}} = 5^{2(3x+4)}$$

$$5^{\frac{1}{2}} = 5^{6x+4}$$

$$\frac{1}{2} = 6x + 4$$

$$2 = 12x + 8$$

$$-6 = 12x$$

$$-\frac{1}{2} = x$$

(RF5.1)

#### Journal

- Ask students to help resolve an argument between three of their friends related to the equation  $\frac{1}{8} = 2^{x-5}$ . Jane claims the equation simplifies to  $-2^3 = 2^{x-5}$ . Betty says the simplified form is  $2^{-3} = 2^{x-5}$ . Robert is convinced that the equation simplifies to  $4^{-2} = 2^{x-5}$ . Ask students who they agree with and why the other two simplifications are incorrect.

(RF5.1)

#### Paper and Pencil

- Ask students to answer the following:
  - Solve  $3(9)^{5x} = 27$ . They should present their solution to the class and discuss the different strategies that were used.
  - Using two different bases, solve  $4^{2x} = 64^{x+1}$ .
  - Solve the following exponential equations:
    - $2^x = 128$
    - $5^{2x+1} = 25$
    - $27^{4x} = 9^{x+1}$
    - $16^{2x+1} = \left(\frac{1}{2}\right)^{x-3}$

(RF5.1)

### Resources/Notes

#### Authorized Resource

*Principles of Mathematics 12*

#### 6.3 Solving Exponential Equations

SB: pp. 352 - 365

TR: pp. 336 - 342

**Note:** Students will do examples similar to example 4 on p.359 of the SB when the concept of logarithms is introduced in the next section.

#### Web Link

The following site contains interactive flash cards related to the law of exponents.

[www.mathmistakes.info/facts/AlgebraFacts/index.html](http://www.mathmistakes.info/facts/AlgebraFacts/index.html)

**Relations and Functions**

**Outcomes**

*Students will be expected to*

RF5 Continued...

**Achievement Indicator:**

RF5.2 *Continued*

**Elaborations – Strategies for Learning and Teaching**

Students should note that the value obtained for  $2^{3.3}$  is closer to 10, so the next estimate should be closer to 3.3 than to 3.4.

3.31	$2^{3.31}$	9.918
3.32	$2^{3.32}$	9.987
3.33	$2^{3.33}$	10.056

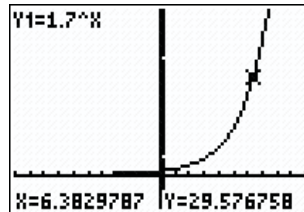
They should reason that the best estimate is  $x \doteq 3.32$  because 9.987 is closer to 10 than is 10.056.

Encourage students to incorporate the method of systematic trial using a variety of bases, including rational bases, although the precision of estimates can vary.

- Graphing Technology

Students could also use graphing technology to determine the solution for an exponential equation. With graphing technology, the solution can be found using the graph or a table of values.

To determine the solution to the equation  $1.7^x = 30$ , for example, the graph of  $y = 1.7^x$  can be used to determine the value of  $x$  that makes the value of the function approximately 30.

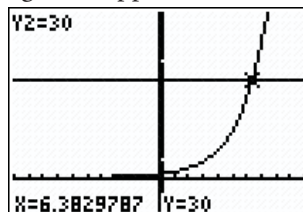


Students could also generate the table of values associated with the equation to find the value of  $x$  that makes the value of the function approximately 30.

X	Y1
6	24.138
6.1	25.453
6.2	26.84
6.3	28.303
6.4	29.845
6.5	31.472
6.6	33.187

X=6.4

Alternatively, they can determine the intersection of the graphs of  $y = 1.7^x$  and  $y = 30$  to get the approximate solution.

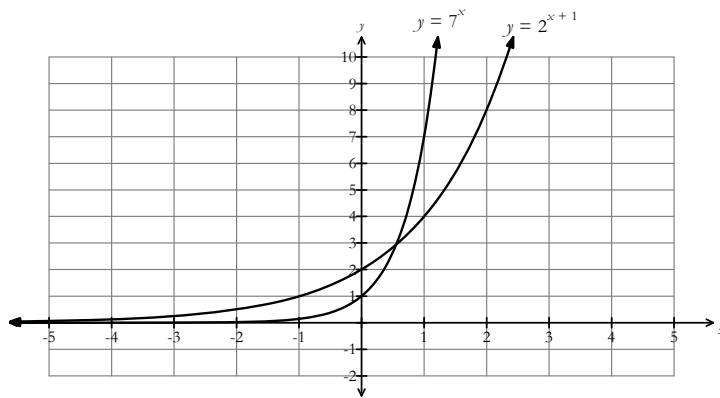


**General Outcome: Develop algebraic and graphical reasoning through the study of relations.**

**Suggested Assessment Strategies**

*Paper and Pencil*

- Ask students to use the graph to estimate the solution to  $7^x = 2^{x+1}$ .



RF5.2)

**Resources/Notes**

**Authorized Resource**

*Principles of Mathematics 12*

**6.3 Solving Exponential Equations**

SB: pp. 352 - 365

TR: pp. 336 - 342

**Relations and Functions**

**Outcomes**

*Students will be expected to*

RF5 Continued...

**Achievement Indicators:**

RF5.3 *Solve problems that involve the application of exponential equations.*

RF5.1 *Continued*

**Elaborations – Strategies for Learning and Teaching**

Remind students that an exponential expression arises when a quantity changes by the same factor for each unit of time. For example, a population doubles every year, a bank account increases by 0.1% each month, or a mass of radioactive substance decreases by  $\frac{1}{2}$  every 462 years, they can expect an exponential expression. Many real world phenomena can be modeled by functions that describe how things grow or decay as time passes.

Introduce students to the half-life exponential function,  $A(t) = A_0\left(\frac{1}{2}\right)^{\frac{t}{b}}$  and discuss the unknown variables. Point out to students that the value of  $b$  is called the half-life since it corresponds to the point on the graph of the function where the value of the function is half its initial value  $A_0$ .

Students should solve problems where the exponential equation is given. Depending on the context of the problem, they should substitute values into the function. Ask students to work through an example such as the following:

- The population of trout growing in a lake can be modeled by the function  $P(t) = 200(2)^{\frac{t}{5}}$  where  $P(t)$  represents the number of trout and  $t$  represents the time in years after the initial count. How long will it take for there to be 6400 trout?

Ask students to create a table of values choosing values of  $x$  and substituting them into the equation. The table allows students to make a connection to the meaning of the values contained in the equation.

$t$	0	1	2	3	4	5
$P(t)$	200	229.74	263.90	303.14	348.22	400

Students should be able to make the following conclusions:

- the value of 200 represents the initial amount of trout in the lake
- the number of trout doubles every 5 years
- they can determine the value of  $t$  when  $P(t) = 6400$  by rewriting the equation with the same base ( $2^5 = 2^{\frac{t}{5}}$ )

Students should be exposed to problems involving half-life and doubling time again in the next unit when they are introduced to logarithms.

**General Outcome: Develop algebraic and graphical reasoning through the study of relations.**

### Suggested Assessment Strategies

#### *Paper and Pencil*

- Small rural water systems are often contaminated with bacteria by animals. Suppose that a water tank is infested with a colony of 14,000 E. coli bacteria. In this tank, the colony doubles in number every 4 days. The number of bacteria present in the tank after  $t$  days can be modeled by the function  $A(t) = 14\,000(2)^{\frac{t}{4}}$ . Ask students to determine the value of  $t$  when  $A(t) = 224\,000$ . What does your answer mean in this context? Explain.

(RF5.1, RF5.3)

- The half-life of a radioactive isotope is 30 hours. The amount of radioactive isotope  $A(t)$ , at time  $t$ , can be modelled by the function  $A(t) = A_0\left(\frac{1}{2}\right)^{\frac{t}{30}}$ . Ask students to determine algebraically how long it will take for a sample of 1792 mg to decay to 56 mg.

(RF5.1, RF5.3)

### Resources/Notes

#### Authorized Resource

#### *Principles of Mathematics 12*

#### 6.3 Solving Exponential Equations

SB: pp. 352 - 365

TR: pp. 336 - 342

**Relations and Functions**

**Outcomes**

*Students will be expected to*  
RF6 Continued...

**Achievement Indicators:**

RF6.4 *Graph data, and determine the exponential function that best approximates the data.*

RF6.5 *Interpret the graph of an exponential function that models a situation, and explain the reasoning.*

RF6.6 *Solve, using technology, a contextual problem that involves data that is best represented by graphs of exponential function and explain the reasoning.*

**Elaborations – Strategies for Learning and Teaching**

Students performed linear, quadratic and cubic regressions when working with polynomial functions. They should continue to use technology to create a scatter plot and determine the equation of the exponential regression function that models the data. The following table shows the household expenditure in Newfoundland and Labrador for the period of 2000 to 2009. Examples of household expenditures include food, shelter, transportation, health care, personal care, education, and recreation.

Year	2000	2001	2002	2003	2004	2005
Household Expenditure	43 501	45 759	46 597	47 944	49 126	52 306

Year	2006	2007	2008	2009
Household Expenditure	53 939	55 007	57 713	57 605

Students may find it beneficial to rewrite the variable to be  $t =$  years since 2000 and then  $H(t) =$  household expenditure  $t$  years after 2000.

$t$	0	1	2	3	4
$H(t)$	43 501	45 759	46 597	47 944	49 126

Guide students through the process using the following directions:

- Construct a scatter plot to display the data.
- Does your graph appear to have an exponential curve pattern? Explain your reasoning.
- Determine the equation of the exponential regression function that models the data.
- Graph the curve of best fit.
- According to this exponential model, what will be the household expenditure for the year 2020? Are there any factors that might cause the actual amount to be different from the amount projected by this model?

**General Outcome: Develop algebraic and graphical reasoning through the study of relations.**

### Suggested Assessment Strategies

#### *Paper and Pencil*

- The following table shows the average undergraduate university tuition fees paid by full time Canadian students.

School Year	Tuition Fee
2004-2005	\$4140
2005-2006	\$4214
2006-2007	\$4400
2007-2008	\$4558
2008-2009	\$4747
2009-2010	\$4942
2010-2011	\$5146
2011-2012	\$5366

Ask students to respond to the following:

- Using graphing technology, construct a scatter plot to display the data.
- Does your graph appear to have an exponential curve pattern?
- Use an exponential regression to define a function that models the data.
- Estimate the tuition fee for the school year 2015-2016.

(RF6.4, RF6.5, RF6.6)

### Resources/Notes

#### Authorized Resource

*Principles of Mathematics 12*

#### 6.4 Modelling Data Using Exponential Functions

SB: pp. 370 - 385

TR: pp. 344 - 352

#### Web Link

The data related to the household expenditures and the tuition fees can be found at [www.statcan.gc.ca](http://www.statcan.gc.ca).

**Relations and Functions**

**Outcomes**

*Students will be expected to*

RF6 Continued...

**Achievement Indicators:**

RF6.4, RF6.5, RF6.6

*Continued*

**Elaborations – Strategies for Learning and Teaching**

Students should be exposed to financial applications involving exponential equations. Introduce students to the concepts of simple interest and compound interest. It is important to build the foundation of simple and compound interest since students will work with these concepts in more detail later in the unit on Finance.

Introduce students to the exponential function that models compound interest,  $A = P(1 + i)^n$  where  $A$  is the future value,  $P$  is the principal,  $i$  represents the interest rate per compounding period (expressed as a decimal) and  $n$  represents the number of compounding periods.

Teachers could use an example, such as a savings account, to compare the relationship between simple and compound interest. Inform students that the principal value of \$1000.00 ( $P$ ) in a savings account earns an annual interest of 5% ( $r = 0.05$ ). Ask students to calculate the total amount in the account at the end of year 1, 2, etc.

*Simple Interest:*

Year ( $t$ )	Total Amount at the End of the Year ( $A$ )
0	\$1000
1	\$1050 $(\$1000 + \$1000(0.05)(1))$
2	\$1100 $(\$1000 + \$1000(0.05)(2))$
3	\$1150 $(\$1000 + \$1000(0.05)(3))$

As students work through the calculations, they should reflect on:

- whether the simple interest earned each year is constant or variable
- the relationship between the number of years, the interest earned each year, and the accumulated interest.

Emphasize to students that simple interest is calculated only in terms of the original amount invested, not on the accumulated interest. When graphing the value of the investment, students should recognize the graph as a series of discrete points in an increasing linear pattern. The discrete points indicate there is no growth between the points, only from the end of one year to the next.

Stress to students that the accumulated value or amount  $A$  is the sum of the principal and the accumulated interest ( $Prt$ ). Introduce them to the simple interest formula  $A = P + Prt$  or  $A = P(1 + rt)$ , where  $P$  is the principal amount,  $t$  represents time in years, and  $r$  represents the interest rate per annum. If students have difficulty applying the simple interest formula, they can create a table of values so they can determine the investment value in increments until they notice a pattern.



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**General Outcome: Develop algebraic and graphical reasoning through the study of relations.**

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**Suggested Assessment Strategies**

*Paper and Pencil*

- Kyle invested his summer earnings of \$5000 at 8% simple interest, paid annually. Ask students to graph the growth of the investment for 6 years using "time (years)" as the domain and "value of the investment" as the range. They should answer the following:
  - (i) What does the shape of the graph tell you about the type of growth? Why is the data discrete?
  - (ii) What do the  $y$ -intercept and slope represent for the investment?
  - (iii) What is the value of the investment after 10 years?

(RF6.4, RF6.5, RF6.6)

**Resources/Notes**

**Authorized Resource**

*Principles of Mathematics 12*

**6.5 Financial Applications**

**Involving Exponential Functions**

SB: pp. 386 - 399

TR: pp. 353 - 364

**Relations and Functions**

**Outcomes**

*Students will be expected to*

**RF5 & RF6 Continued...**

**Achievement Indicators:**

RF6.4, RF6.5, RF6.6, RF5.3  
*Continued*

**Elaborations – Strategies for Learning and Teaching**

*Compound Interest:*

Guide students as they set up the table for the compound interest investment and work through the calculations. Ask students if they notice any patterns within the table of values. Do they notice, for example, that the accumulated interest and the value of the investment do not grow by a constant amount as they do with simple interest.

Year ( $t$ )	Amount of Annual Interest	Total Amount at the End of the Year ( $A$ )
0		\$1000
1	$1000 \times 0.05 = 50$	\$1050 $(1000(1.05)^1)$
2	$1050 \times 0.05 = 52.50$	\$1102.50 $(1000(1.05)^2)$
3	$1102.50 \times 0.05 = 55.13$	\$1157.63 $(1000(1.05)^3)$
4	$1157.63 \times 0.05 = 57.88$	\$1215.51 $(1000(1.05)^4)$

Students should be able to identify that compound interest is determined by applying the interest rate to the sum of the principal and any accumulated interest.

The formula for compound interest is different than simple interest. Ask students to create a scatter plot and perform an exponential regression to model the investment, resulting in the equation  $y = 1000(1.05)^x$ .

Students should compare this exponential regression to the compound interest formula  $A = P(1 + i)^n$ . Inform students that the variables  $i$  and  $n$  represent interest rate and time, but not in the same way as the simple interest formula. The variable  $i$  is the interest rate per compounding period, and the variable  $n$  is the number of compounding periods. Since the interest is compounded once a year,  $i = \frac{0.05}{1}$ , the exponential regression  $y = 1000(1.05)^x$  can be written as  $P = 1000(1.05)^n$ .

Up to this point, students have only been exposed to investments earning compound interest once per year. They should also work with investments that have daily, weekly, monthly, quarterly, semi-annually, or annually compounding periods. Ask students to write an exponential equation if \$1000, for example, is invested at 6% compounded monthly. Students should reason that if the interest is compounded every month, annual interest is paid 12 times per year (i.e., the interest rate per compounding period is  $i = \frac{\text{annual rate}}{12}$ ). The compound interest formula is defined as  $y = 1000(1.005)^n$  where  $n$  is the number of months.

**General Outcome: Develop algebraic and graphical reasoning through the study of relations.**

**Suggested Assessment Strategies**

*Paper and Pencil*

- Ask students to answer the following:
  - Consider the following statements:
    - Emily invested \$3000 for a term of 5 years with a simple interest rate of 4% per year, paid annually.
    - Zachary invested \$3000 for a term of 5 years with a compound interest rate of 4% per year, paid annually.
  - (i) Using technology, graph the investments. What do you notice?
  - (ii) Which investment will result in a greater future value? Explain your reasoning.

(RF6.4, RF6.5, RF6.6)
- \$3000 was invested at 6% per year compounded monthly. Ask students to answer the following:
  - (i) Write the exponential function in the form  $A = P(1 + i)^n$ .
  - (ii) Create a table of values of the investment at the end of four months. Use these values to create the equation of the exponential regression function that models the investment.
  - (iii) What do you notice about the equations in (i) and (ii) ?
  - (iv) What will be the future value of the investment after 4 years?

(RF6.4, RF6.5, RF6.6)
- \$2000 is invested at 6% per year compounded semi-annually. Carol defined the exponential function as  $A = 2000(1.03)^n$  where  $n$  is the number of 6 month periods. Is Carol's reasoning correct? Why or why not?

(RF5.3)

**Resources/Notes**

**Authorized Resource**

*Principles of Mathematics 12*

**6.5 Financial Applications  
Involving Exponential Functions**

SB: pp. 386 - 399

TR: pp. 353 - 364

**Relations and Functions**

**Outcomes**

*Students will be expected to*  
 RF5 & RF6 Continued...

**Achievement Indicators:**

RF6.4, RF6.5, RF6.6, RF5.3  
*Continued*

**Elaborations – Strategies for Learning and Teaching**

Students should solve problems where they apply the equation  $A = P(1 + i)^n$  to determine the future value of the investment after a certain time, assuming the interest rate remains the same over the entire time. Solving for the unknowns  $n$  or  $i$ , will be explored once students complete their work with logarithms in the next unit.

Up to this point, students have solved a variety of problems involving appreciation. They should also be introduced to problems where a value depreciates. Ask students to construct a table of values to represent the following example:

- An automobile that originally costs \$24 000 loses one-fifth of its value each year. What is the value after 6 years?

Students could perform an exponential regression resulting in the equation  $y = 24000(0.8)^x$ , where  $x$  represents the number of years after purchase and  $y$  represents the value of the car. Ask students what the value 0.8 represents in the context of the problem. Students should be thinking that if the car depreciates by 20% each year, the car's value is 80% of what it was the year before (i.e., depreciation rate = 100% - 20% = 80%). The base of the exponential is a decimal that can be written as a percent. Inform students that they will solve these problems again, using an algebraic approach, in the next unit when logarithms are introduced.

Exponential functions also apply to loan payments. Whether someone can afford a loan depends on whether he/she can afford the periodic payment (commonly a monthly payment period).

- Mary needs to borrow \$20 000 to buy a car. The bank is charging 7.5% annual interest rate compounded monthly. Her monthly loan payment is \$400.76. The time required to pay off the loan can be represented by  $(1.00625)^{-n} = 0.6880926$ , where  $n$  is the number of months. Determine how long it would take her to pay off the loan and how much interest she will have to pay on the loan.

When calculating the time to pay off a loan, students will not be expected to develop the formula. The formula for this application should be provided in the problem. Students can solve this problem using graphing technology or they can solve it algebraically when they are introduced to logarithms in the next unit.

**General Outcome: Develop algebraic and graphical reasoning through the study of relations.**

**Suggested Assessment Strategies**

*Paper and Pencil*

- Joe invests \$5000 into a high interest savings bond that has an annual interest rate of 9% compounded monthly. Ask students to answer the following:
  - (i) Write the exponential equation  $A = P(1 + i)^n$  that represents the situation.
  - (ii) How much will the bond be worth after 1 year?

(RF5.3)

- \$1000 is invested at 8.2% per year for 5 years. Using the equation  $A = P(1 + i)^n$ , ask students to determine the account balance if it is compounded annually, quarterly, monthly and daily. Would it make more of a difference if the interest is accrued for 25 years? Explain your reasoning.

(RF5.3)

- \$2000 is invested for three years that has an annual interest rate of 9% compounded monthly. Lucas solved the following equation  $A = 2000(1.0075)^3$ . Ask students to identify the error that Lucas made? They should correct the error and solve the problem.

(RF5.3)

*Journal*

- Ask students to choose between two investment options and justify their choice: earning 12% interest per year compounded annually or 12% interest per year compounded monthly.

(RF5.3)

**Resources/Notes**

**Authorized Resource**

*Principles of Mathematics 12*

**6.5 Financial Applications  
Involving Exponential Functions**

SB: pp. 386 - 399

TR: pp. 353 - 364

**Note:**

Students will do Example 3 on p.392 of the SB after they complete the unit on logarithms.



# Logarithmic Functions

Suggested Time: 13 Hours

# Unit Overview

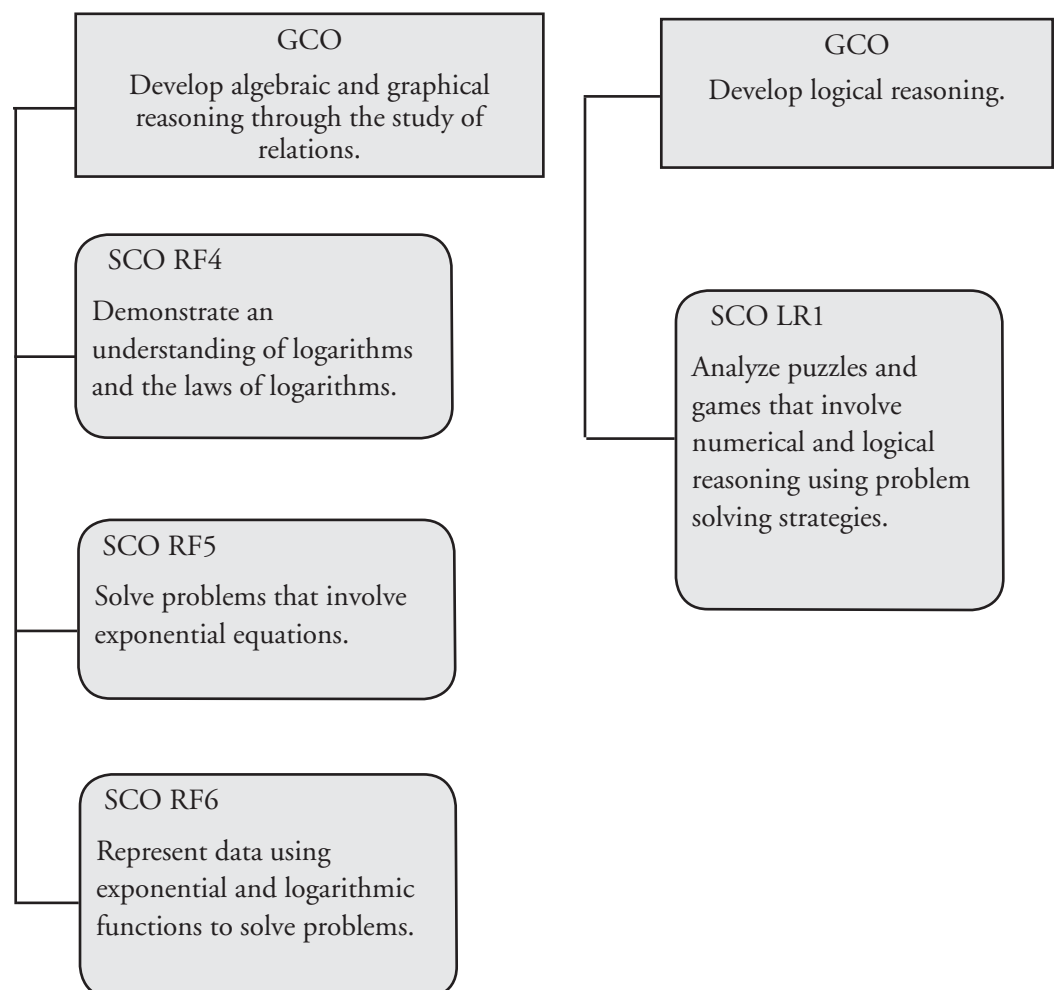
## Focus and Context

In the previous unit, students worked with exponential equations where the bases were powers of one another. They will now solve exponential equations with different bases.

Students will be introduced to logarithms, how to represent them, and how to use them to model and solve problems. Using the laws of logarithms, students simplify and evaluate expressions.

Using graphing technology, students will enter data and create a model using logarithmic regression. Predictions are made by reading the values from the curve of best fit on a scatter plot or by using the equation of the logarithmic regression function.

## Outcomes Framework





Mathematical Processes

- [C] Communication
- [PS] Problem Solving
- [CN] Connections
- [R] Reasoning
- [ME] Mental Mathematics and Estimation
- [T] Technology
- [V] Visualization

SCO Continuum

Mathematics 1201	Mathematics 2201	Mathematics 3201
Algebra and Number		Relations and Functions
AN3 Demonstrate an understanding of powers with integral and rational exponents. [C, CN, PS, R]	not addressed	RF4 Demonstrate an understanding of logarithms and the laws of logarithms. [C, CN, ME, R]  RF5 Solve problems that involve exponential equations. [C, CN, PS, R, T]  RF6 Represent data using exponential and logarithmic functions to solve problems. [C, CN, PS, T, V]
Number and Logic		Logical Reasoning
not addressed	NL2 Analyze puzzles and games that involve spatial reasoning, using problem solving strategies. [CN, PS, R, V]	LR1 Analyze puzzles and games that involve numerical and logical reasoning using problem solving strategies. [CN, ME, PS, R]

## Relations and Functions

### Outcomes

Students will be expected to

**RF6 Represent data, using exponential and logarithmic functions, to solve problems.**

[C, CN, PS, T, V]

### Achievement Indicators:

RF6.7 Describe, orally and in written form, the characteristics of a logarithmic function by analyzing its graph.

RF6.8 Describe, orally and in written form, the characteristics of a logarithmic function by analyzing its equation.

### Elaborations – Strategies for Learning and Teaching

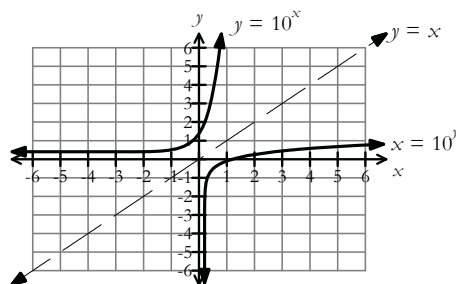
This outcome was addressed in the previous unit in relation to the exponential function of the form  $y = a(b)^x$ , where  $b > 0$ ,  $b \neq 1$ , and  $a > 0$ . In this unit, students will be introduced to the logarithmic function of the form  $y = a \log_b x$  where  $b > 0$ ,  $b \neq 1$ ,  $a > 0$ , and  $a$  and  $b$  are real numbers. They will compare the graphs of exponential functions and logarithmic functions, and identify characteristics with emphasis on domain and range, intercepts, and end behaviour. Students have not been exposed to the concept of inverse in previous courses. They will compare exponential and logarithmic functions as reflections about the line  $y = x$ .

Students should explicitly graph logarithmic functions with base 10 and base  $e$ . Later in this unit, they will simplify logarithmic expressions with other bases.

It may be beneficial for students to first investigate the characteristics of logarithmic functions with base 10. Using graphing technology, ask students to graph  $y = 10^x$  and the line  $y = x$ .

Teachers should ask the following questions to help students make the connection between exponential and logarithmic functions.

- How can a reflection of  $y = 10^x$  be drawn about the line  $y = x$ ? Why is it necessary to interchange the  $x$  and  $y$ -variables?
- Use the table of values of  $y = 10^x$  to generate the table for  $x = 10^y$ . How are the  $x$ - and  $y$ -values affected?
- What are the coordinates of the new function?
- What are the similarities and differences between the two functions with reference to domain, range, intercepts, and end behaviour?



$y = 10^x$	$x = 10^y$
$(-2, \frac{1}{100})$	$(\frac{1}{100}, -2)$
$(-1, \frac{1}{10})$	$(\frac{1}{10}, -1)$
$(0, 1)$	$(1, 0)$
$(1, 10)$	$(10, 1)$
$(2, 100)$	$(100, 2)$

Teachers should inform students that:

- the equation of the reflected graph can be written as a common logarithmic function  $y = \log_{10} x$  (i.e.,  $y = \log x$ ).
- the  $x$  and  $y$ -values on the exponential curve can be switched to find the coordinates of the points on the logarithmic curve.
- logarithms are a different form of an exponential statement. The logarithm  $y = \log_{10} x$  is asking, *What exponent,  $y$ , is need so that  $10^y = x$ .*

Later in this unit, students will express a logarithmic equation as an exponential equation and vice versa in more depth.

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**General Outcome: Develop algebraic and graphical reasoning through the study of relations.**

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**Suggested Assessment Strategies**

*Journal*

- Mary evaluated  $\log(-3.24)$  on her calculator and an error message was displayed. Ask students to explain why an error message occurred.

(RF6.7, RF6.8)

**Resources/Notes**

**Authorized Resource**

*Principles of Mathematics 12*

**7.1 Characteristics of Logarithmic Functions with Base 10 and Base  $e$**

Student Book (SB): pp. 412 - 425  
Teacher Resource (TR): pp. 390 - 396

## Relations and Functions

### Outcomes

*Students will be expected to*

RF6 Continued...

### Achievement Indicators:

RF6.7, RF6.8 *Continued*

### Elaborations – Strategies for Learning and Teaching

Students should notice the graph of the logarithmic curve ( $y = \log_{10} x$ ) is a reflection of the exponential curve ( $y = 10^x$ ). They should summarize the characteristics of the graphs using a table similar to the one shown.

	Exponential	Logarithmic
Domain	all reals	positive $x$ values
Range	positive $y$ values	all reals
$y$ -intercept	one $y$ -intercept (0, 1)	no $y$ -intercept
$x$ -intercept	no $x$ -intercept	one $x$ -intercept (1, 0)
increasing/ decreasing	increasing from Quadrant II to Quadrant I	increasing from Quadrant IV to Quadrant I
end behaviour	as $x \rightarrow \infty$ , $y \rightarrow \infty$ as $x \rightarrow -\infty$ , $y \rightarrow 0$	as $x \rightarrow \infty$ , $y \rightarrow \infty$ as $x \rightarrow 0$ on the right, $y \rightarrow -\infty$

Students should then investigate the effect of changing the value of  $a$  by comparing logarithmic functions  $y = \log_{10} x$  and  $y = a \log_{10} x$  where  $a \neq 0$ . Use graphing technology to graph and analyze logarithmic functions, such as  $y = \log_{10} x$ ,  $y = 4 \log_{10} x$ , and  $y = -4 \log_{10} x$ . They should be able to answer the following:

- What is the impact on the graph of the function if  $a > 0$ ? if  $a < 0$ ?
- Does  $a$  affect the  $x$ -coordinate or the  $y$ -coordinate? Is this a vertical transformation or a horizontal transformation?
- Which point is easily identified from the graph?
- Which characteristics of the graphs of logarithmic functions differ from the characteristics of the graphs of exponential functions?

Students should recognize when  $a > 0$ , the  $y$ -values increase as the  $x$ -values increase. This is an increasing function from Quadrant IV to Quadrant I. If  $a < 0$ ,  $y$ -values decrease as the  $x$ -values increase. This is a decreasing function from Quadrant I to Quadrant IV. They should recognize that logarithmic functions do not have a  $y$ -intercept but do have a restricted domain (i.e.,  $x > 0$ ).

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**General Outcome: Develop algebraic and graphical reasoning through the study of relations.**

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**Suggested Assessment Strategies**

*Journal*

- Ask students to explain, using the graph of  $y = \log x$ , why they cannot evaluate  $\log(-3)$  and  $\log(0)$ .  
(RF6.7, RF6.8)

**Resources/Notes**

**Authorized Resource**

*Principles of Mathematics 12*

7.1 Characteristics of Logarithmic Functions with Base 10 and Base  $e$

Student Book (SB): pp. 412 - 425  
Teacher Resource (TR): pp. 390 - 396

## Relations and Functions

### Outcomes

*Students will be expected to*

RF6 Continued...

#### Achievement Indicators:

RF6.7, RF6.8 *Continued*

RF6.9 *Match equations in a given set to their corresponding graphs.*

### Elaborations—Strategies for Learning and Teaching

Students should compare the natural logarithmic function  $y = \log_e x$  (i.e.,  $y = \ln x$ ) and the exponential function  $y = e^x$ . Discuss with students that the value of  $e$  is an irrational number. Using graphing technology, they should observe that the graph of  $y = \ln x$  is a reflection of the graph of  $y = e^x$  about the line  $y = x$ . The visual aid should also help students realize that changing the base from  $b = 10$  to  $b = e$  has no effect on the characteristics of the graphs. Students will work in more detail with natural logarithms when they perform logarithmic regressions on a set of data later in this unit.

Students should match equations of exponential and logarithmic functions in a given set to their corresponding graphs. As students distinguish between logarithmic functions of the form  $y = a \log x$  or  $y = a \ln x$  and exponential functions  $y = a(b)^x$ , they should reflect on the following features:

- exponential function either extends from Quadrant II to Quadrant I or vice versa
- logarithmic function either extends from Quadrant IV to Quadrant I or vice versa
- the value of  $a$  in the logarithmic function is used to determine if the function is increasing or decreasing
- the value of  $b$  in exponential functions is used to determine if the function is increasing or decreasing
- the value of  $a$  in the exponential function is used to determine the  $y$ -intercept

**General Outcome: Develop algebraic and graphical reasoning through the study of relations.**

**Suggested Assessment Strategies**

*Performance*

- Ask students to participate in the activity *Speed Match*. Place desks in pairs and distribute one cue card to each student containing a logarithmic graph or a logarithmic equation. Students will then be given a time frame to ask their partner questions to determine if they have found their match. The person with the graph will ask questions and their partner may only answer yes or no. When the time is up the person asking the questions will move on to the next person. This will continue until they find their match. (Variation: The person with the equation can ask the questions.)

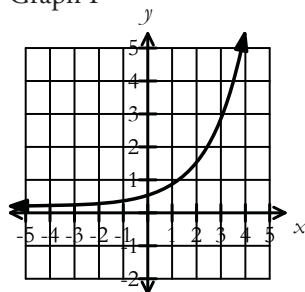
(RF6.9)

*Paper and Pencil*

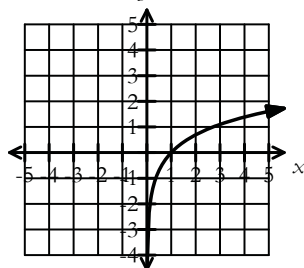
- Ask students to match the equations with the graphs.

Equation I	$y = 3\left(\frac{1}{2}\right)^x$	Equation II	$y = \frac{1}{3}(2)^x$
Equation III	$y = \ln x$	Equation IV	$y = -2\ln x$

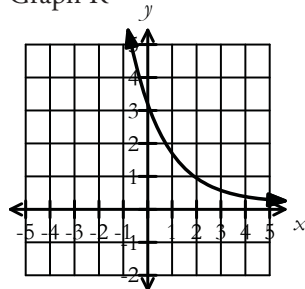
Graph P



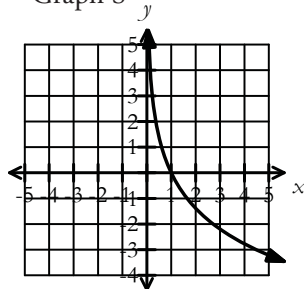
Graph Q



Graph R



Graph S



(RF6.9)

**Resources/Notes**

**Authorized Resource**

*Principles of Mathematics 12*

7.1 Characteristics of Logarithmic Functions with Base 10 and Base  $e$

SB: pp. 412 - 425

TR: pp. 390 - 396

## Relations and Functions

### Outcomes

*Students will be expected to*

**RF4 Demonstrate an understanding of logarithms and the laws of logarithms.**

[C, CN, ME, R]

### Achievement Indicators:

RF4.1 *Express a logarithmic equation as an exponential equation and vice versa.*

RF4.2 *Determine the value of a logarithmic expression, such as  $\log_2 8$ , without technology.*

### Elaborations – Strategies for Learning and Teaching

Students should express a logarithmic equation as an exponential equation and vice versa. They evaluate a logarithmic expression by writing the expression in exponential form.

The laws of logarithms should be developed using both numerical examples and the exponent laws. Students write an equivalent expression for a logarithmic expression using the laws of logarithms. Logarithms are used when solving exponential equations, where the bases are not powers of one another. Students will not, however, be expected to solve logarithmic equations using the laws of logarithms (i.e.,  $\log_2(4x-1) - \log_2(2x+1)=3$ ).

Students graphed logarithmic functions restricted to bases 10 and  $e$ . When evaluating logarithmic expressions students should work with a base  $b$  where  $b > 0$  and  $b \neq 1$ .

Given a statement in logarithmic form, students should be able to write it in exponential form, and vice versa. Teachers should remind students that logarithms are a different form of an exponential statement. The statement  $3^2 = 9$ , for example, can be written as  $\log_3 9 = 2$ . Ask students what they notice about the base of the exponent and the base of the logarithm. Ask students to think about  $\log_b x = y$  as *What exponent is needed so that,  $b^y = x$ .*

Students should determine the value of a logarithmic expression without the use of technology. When evaluating  $\log_2 8$ , for example, it may be helpful to write the expression as  $\log_2 8 = x$ . Students can evaluate this logarithmic equation by writing the equivalent exponential form,  $2^x = 8$ . When interchanging between the two forms, encourage students to ask themselves the following question, "The number 2 raised to what exponent is 8?" This type of questioning should help them verify the values of logarithmic expressions. When the value of a logarithm is a rational number, such as  $\log_{64} 4$  and  $\log_{\frac{1}{2}} 32$ , students should continue to determine the exact value without the use of technology.



**General Outcome: Develop algebraic and graphical reasoning through the study of relations.**

### Suggested Assessment Strategies

#### *Performance*

- Quiz-Quiz-Trade: Each student is given a card with a problem. The answer is written on the back of the card. In groups of two, partner A asks the question and partner B answers. They switch roles and repeat. Students move around the classroom until every student has had a chance to solve all the problems. Sample cards are shown below.

Express $71 = e^x$ in logarithmic form.	Evaluate the logarithmic expression $\log_3\left(\frac{1}{81}\right)$ .
Evaluate the logarithmic expression $\log_9 27$ .	Evaluate the following expression: $\log_2 16 + \log_3 27$
Evaluate the following expression: $\log_{\frac{1}{2}} 16 \div \log_{\frac{1}{3}} 27$	

(RF4.1, RF4.2)

### Resources/Notes

#### Authorized Resource

*Principles of Mathematics 12*

#### 7.2 Evaluating Logarithmic Expressions

SB: pp. 426 - 438

TR: pp. 397 - 402

## Relations and Functions

### Outcomes

*Students will be expected to*

RF4 Continued...

#### Achievement Indicators:

RF4.3 *Develop the laws of logarithms, using numeric examples and the exponent laws.*

### Elaborations—Strategies for Learning and Teaching

In Mathematics 1201, students applied the exponent laws to expressions with rational and variable bases, and integral and rational exponents (AN3). They worked with the following properties:

- $a^n \times a^m = a^{n+m}$
- $a^n \div a^m = a^{n-m}$
- $(a^n)^m = a^{nm}$

Teachers should introduce the logarithmic rules where  $b > 0$ ,  $m > 0$ ,  $n > 0$  and  $b \neq 1$  where  $b$ ,  $m$ , and  $n \in R$ , and ask students if they notice any similarities between the exponent properties and the laws of logarithms.

- Product Law             $\log_b(m \times n) = \log_b m + \log_b n$
- Quotient Law            $\log_b(m \div n) = \log_b m - \log_b n$
- Power Law:              $\log_b m^n = n \log_b m$

Teachers should show several numerical examples to allow students to discover the laws of logarithms inductively. To develop the product law, for example, a procedure similar to the following could be used:

- begin with the exponential equations:  $8 = 2^3$  and  $16 = 2^4$
- convert each equation to logarithmic form:  $\log_2 8 = 3$  and  $\log_2 16 = 4$
- rewrite  $8 \times 16$  in exponential form:  $8 \times 16 = 2^{3+4}$
- convert this equation back to logarithmic form:  $\log_2(8 \times 16) = 3 + 4$
- substitute  $\log_2 8$  in for the value of 3 and  $\log_2 16$  in for the value of 4:  
 $\log_2(8 \times 16) = \log_2 8 + \log_2 16$

Verification may now be illustrated by checking that the left hand side of the equation equals the right hand side of the equation. A similar exercise should be used to develop the quotient rule.

A common students error occurs when students apply the product law incorrectly. They may write, for example,  $\log 5 = \log 3 + \log 2$ . Ask students to use a calculator to evaluate their answers to see if they are equivalent. Evaluating the left side of the equation and then the right side will reveal that the values are not equivalent.

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**General Outcome: Develop algebraic and graphical reasoning through the study of relations.**

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**Suggested Assessment Strategies**

*Paper and Pencil*

- Ask students to verify the following using the logarithm laws:

- (i)  $\log_3 27 = \log_3 9 + \log_3 3$
- (ii)  $\log_5 25 = \log_5 125 - \log_5 5$
- (iii)  $\log_2 64 = 6\log_2 2$

(RF4.3)

**Resources/Notes**

**Authorized Resource**

*Principles of Mathematics 12*

**7.3 Laws of Logarithms**

SB: pp: 442-448

TR: pp: 404-410

## Relations and Functions

### Outcomes

*Students will be expected to*

RF4 Continued...

#### Achievement Indicators:

RF4.3 *Continued*

RF4.4 *Determine an equivalent expression for a logarithmic expression by applying the laws of logarithms.*

### Elaborations—Strategies for Learning and Teaching

Work through a numerical example with students to help them understand the reasoning behind the power rule of logarithms. Guide students using the following prompts to develop the law

$\log_b m^n = n \log_b m$ :

- start with the exponential  $8 = 2^3$  (i.e.,  $\log_2 8 = 3$ )
- raise both sides of the equation to a power  $p$ :  $8^p = (2^3)^p$
- simplify the equation:  $8^p = (2^{3p})$
- convert this equation in logarithmic form:  $\log_2 8^p = 3p$
- substitute  $\log_2 8$  in for the value of 3:  $\log_2 (8^4) = (\log_2 8) \times (4)$
- Rewrite:  $\log_2 (8^4) = 4 \log_2 8$

Students should validate the laws by evaluating specific numerical examples such as:

$$\begin{array}{ll} \log_3(9 \times 27) = \log_3 9 + \log_3 27 & \log_2(256 \div 32) = \log_2 256 - \log_2 32 \\ \log_3(243) = \log_3 9 + \log_3 27 & \log_2(8) = \log_2 256 - \log_2 32 \\ 5 = 2 + 3 & 3 = 8 - 5 \\ 5 = 5 & 3 = 3 \end{array}$$

$$\begin{array}{l} \log_2 4^3 = 3 \log_2 4 \\ \log_2 64 = 3 \log_2 4 \\ 6 = 3(2) \\ 6 = 6 \end{array}$$

Students should be able to apply the laws of logarithms to write a logarithmic expression as a single logarithm in simplest form. When simplifying  $2 \log_3 4 + \log_3 5$ , for example, students should write  $\log_3 80$ .

Students should also be exposed to questions where they are required to write a logarithmic expression as a single logarithm and simplify if necessary. Students may initially write  $\log_6 72 - \log_6 2$  as  $\log_6 36$ . Ask students can this expression be simplified further. Ask them if 36 can be written as a power of 6. Examples similar to this may lead to a discussion of the general case  $\log_a a^n = n$  and the specific case of  $\log_a a = 1$ .

## General Outcome: Develop algebraic and graphical reasoning through the study of relations.

### Suggested Assessment Strategies

#### *Paper and Pencil*

- In groups, ask students to solve a variety of selected response questions. The groups will then trade their questions with another group. Ask each group to verify the solutions. If it is incorrect, they should explain why some student responses may have led to certain answers. Samples are shown below:

Simplify:  $\log_5 36 + 2\log_5 3$

Response A:  $\log_5 36 + \log_5 3^2$

$$\log_5 36 + \log_5 6$$

$$\log_5 (36 \times 6)$$

$$\log_5 324$$

Error: student wrote  $3^2$  as 6

Response B:  $\log_5 36 + \log_5 3^2$

$$\log_5 36 + \log_5 9$$

$$\log_5 (36 \div 9)$$

$$\log_5 4$$

Error: student wrote the sum of the two logarithms as the logarithm of a quotient

(RF4.4)

#### *Performance*

- Ask students to work in groups of two for this activity. Each group should be given a deck of cards: (i) 12 cards with a logarithmic expression, (ii) 12 cards with the simplified answer to each question respectively, (iii) 1 YOU WIN card. The dealer shuffles all the cards and deals them out. Students match the logarithmic expression with the simplified form (or the correct answer). Remove the matches and place them face up on the table. Next, players draw a card from their partner. They should locate the matching expression (or answer) and add it to their pairs that are face up on the table. Students take turns drawing cards from their partner's hands. The game continues until all matches are made. One player will have the YOU WIN card.

(RF4.4)

#### *Observation*

- Ask students to identify the error(s) in the solution below and provide the correct solution.

$$\frac{1}{2}\log_2 64 - (2\log_2 6 - \frac{1}{2}\log_2 81)$$

$$= \log_2 64^{\frac{1}{2}} - (\log_2 6^2 - \log_2 81^{\frac{1}{2}})$$

$$= \log_2 8 - (\log_2 12 - \log_2 9)$$

$$= \log_2 8 - (\log_2 3)$$

$$= \log_2 5$$

(RF4.4)

### Resources/Notes

#### Authorized Resource

*Principles of Mathematics 12*

7.3 Laws of Logarithms

SB: pp. 442 - 448

TR: pp. 404 - 410

## Relations and Functions

### Outcomes

*Students will be expected to*

**RF5** Solve problems that involve exponential equations.

[C, CN, PS, R, T]

### Achievement Indicator:

RF5.4 *Solve problems that involve logarithmic scales, such as Richter scale and the pH scale.*

### Elaborations – Strategies for Learning and Teaching

Students should solve problems involving logarithmic scales such as the Richter scale (used to measure the magnitude of an earthquake), the pH scale (used to measure the acidity of a solution) and the decibel scale (used to measure sound level). Students will not be expected to develop formulas but will be given the formula for each pH scale when required.

Students may be familiar with pH scales in science and chemistry courses. The pH scale is used to measure the acidity of a solution.

The pH of a solution is determined using the equation  $p(x) = -\log x$  where  $x$  is the concentration of hydrogen ions measured in moles per litre (mol/L). This scale ranges from 0 to 14 with the lower numbers being acidic and the higher numbers being basic. A value where the pH = 7 is considered neutral. The scale is a logarithmic scale with one unit of increase in pH resulting in a 10 fold decrease in acidity. Another way to consider this would be a one unit increase in pH results in a 10 fold increase in basicity.

The magnitude of an earthquake,  $y$ , can be determined using  $y = \log x$ , where  $x$  is the amplitude of the vibrations measured using a seismograph. An increase of one unit in magnitude results in a 10 fold increase in the amplitude. This topic lends itself to the incorporation of current events with the inclusion and comparison of a variety of earthquakes.

Sound levels are measured in decibels using the function  $\beta = 10 (\log I + 12)$ , where  $\beta$  is the sound level in decibels (dB) and  $I$  is the sound intensity measured in watts per metre squared ( $\text{W}/\text{m}^2$ ).

Students should be exposed to problems where they are asked to compare the intensity of different earthquakes, compare the sound intensity of different events, or compare the acidity of different solutions.

This could be a good opportunity for students to measure audio volume in the environment around them. They can use an application for a smartphone, for example, to show the approximate decibel level wherever they and their smartphone are located. Although quite accurate, the application is mainly a tool for detecting noise levels in casual settings.

**General Outcome: Develop algebraic and graphical reasoning through the study of relations.**

### Suggested Assessment Strategies

*Paper and Pencil*

- Ask students to answer the following questions using the table below.

Location and Date	Magnitude
Chernobyl, 1987	4
Haiti, January 12, 2012	7
Northern Italy, May 20, 2012	6

- How many times as intense was the earthquake in Haiti compared to the one in Chernobyl?
- How many times as intense was the earthquake in Haiti compared to the one in Northern Italy?
- How many times as intense was the earthquake in Northern Italy compared to the one in Chernobyl?
- If a recent earthquake was half as intense as the one in Haiti what would be the approximate magnitude?

(RF5.4)

### Resources/Notes

**Authorized Resource**

*Principles of Mathematics 12*

**7.2 Evaluating Logarithmic Expressions**

SB: pp. 426 - 438

TR: pp. 397 - 402

## Relations and Functions

### Outcomes

*Students will be expected to*

RF5 Continued...

#### Achievement Indicator:

RF5.2 Determine the solution of an exponential equation in which the bases are not powers of one another; e.g.,  $2^{x-1} = 3^{x+1}$ .

### Elaborations—Strategies for Learning and Teaching

To solve an equation such as  $2^x = 8$ , students can rewrite the powers with the same base  $2^x = 2^3$  resulting in  $x = 3$ . Alternatively, logarithms can be used. Students can solve an exponential equation algebraically by taking logarithms of both sides of the equation. They can then apply the power rule for logarithms to solve for the unknown.

$$\begin{aligned} 2^x &= 8 \\ \log 2^x &= \log 8 \\ x \log 2 &= \log 8 \\ x &= \frac{\log 8}{\log 2} \\ x &= 3 \end{aligned}$$

When taking the logarithm of both sides of an equation, ask students if they can use any base logarithm and why is base 10 particularly useful. Ask students which strategy, creating like bases or logarithms, they would use and why.

Students should be exposed to exponentials in which the bases are not powers of one another. Ask students to solve an equation similar to  $2^x = 3^x$  and then progress to an equation such as  $2^{x-1} = 3^{x+1}$ . When solving problems, they should be able to identify the difference between an exact and an approximate solution for an equation.

$$\begin{aligned} \log 2^{x-1} &= \log 3^{x+1} \\ (x-1) \log 2 &= (x+1) \log 3 \\ x \log 2 - \log 2 &= x \log 3 + \log 3 \\ x \log 2 - x \log 3 &= \log 3 + \log 2 \\ x(\log 2 - \log 3) &= \log 3 + \log 2 \\ x &= \frac{\log 2 + \log 3}{\log 2 - \log 3} \\ x &\approx -4.3 \end{aligned}$$

When the exponent is a binomial and the power law is applied, students sometimes forget to apply the distributive property. Ensure they place brackets around the exponent when applying the power law. Students may also have difficulty solving an equation with a variable on both sides. Remind them to collect like terms and factor out the common factor  $x$ .

A common student error occurs when students solve equations similar to  $4(3^{2x}) = 24$ . Discuss with students why  $4(3^{2x})$  cannot be expressed as  $12^{2x}$ . Students should divide both sides by 4 in order to isolate the base with the exponent (i.e.,  $3^{2x} = 6$ ).



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**General Outcome: Develop algebraic and graphical reasoning through the study of relations.**

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**Suggested Assessment Strategies**

*Paper and Pencil*

- Ask students to evaluate the following using the laws of logarithms:

(i)  $3\log_6(2) + \log_6(27)$

(ii)  $\log_5(2.5) + 2\log_5(10) - \log_5(2)$

(RF4.1, RF4.4, RF5.2)

*Observation*

- Ask students to outline how they would teach a classmate the relationship between exponential and logarithmic functions, the restrictions associated with exponential and logarithmic functions, and simplifying logarithmic expressions.

As teachers observe students' work, look for the following guidelines:

- (i) include general steps to follow
- (ii) use mathematical terms correctly
- (iii) provide clear examples
- (iv) describe common errors and how they can be avoided

(RF4.1, RF4.4, RF5.2)

**Resources/Notes**

**Authorized Resource**

*Principles of Mathematics 12*

7.4 Solving Exponential Equations Using Logarithms

SB: pp. 449 - 459

TR: pp. 411 - 419

## Relations and Functions

### Outcomes

*Students will be expected to*

RF4 Continued...

### Achievement Indicators:

RF4.5 *Determine the approximate value of a logarithmic expression, such as  $\log_2 9$ , with technology.*

### Elaborations – Strategies for Learning and Teaching

Earlier in this unit, students determined the exact value of a logarithmic expression without the use of technology. The examples were limited to creating exponential equations where the bases could be rewritten as the same power (i.e.,  $\log_3 27 = x$ ). Students should now work with examples where they can determine the exact or approximate solution regardless of the base of the logarithmic expression.

When determining the value of  $\log_2 9$ , for example, ask students to first use benchmarks in their estimating. The value of  $\log_2 8 = 3$  and  $\log_2 16 = 4$ . They should notice that  $\log_2 9$  is closer to  $\log_2 8$  than  $\log_2 16$ , therefore  $\log_2 9$  is a little greater than 3 (i.e., 3.2).

By setting the expression equal to  $x$ , students can take the logarithm of both sides and solve for the unknown.

$$\begin{aligned}\log_2 9 &= x \\ 2^x &= 9 \\ \log 2^x &= \log 9 \\ x \log 2 &= \log 9 \\ x &= \frac{\log 9}{\log 2} \approx 3.17\end{aligned}$$

As students work through this example and several others, they may inductively predict a relationship between the original equation and the solution.

$$\log_2 9 = x \longrightarrow x = \frac{\log 9}{\log 2}$$

Students can use this property  $\log_a x = \frac{\log x}{\log a}$ , where  $a \neq 1$ ,  $x > 0$ , to determine the logarithm of any base.

**General Outcome: Develop algebraic and graphical reasoning through the study of relations.**

### Suggested Assessment Strategies

#### Performance

- For the activity *Choose a Partner*, half the students are given logarithmic expressions and the other half are given the associated simplified logarithmic expression. Students move around the classroom attempting to match equivalent expressions. Once they are in pairs, they work together to evaluate the simplified expression. The final step for the pair would be to find their solution from a list of answers that are posted on the wall.

$3\log_5 25 + \log_3 2$	$6 + \log_3 2$	6.63
$2\log_4 8 + 3\log_4 \left(\frac{1}{2}\right)$	$3 + \log\left(\frac{6}{81}\right)$	3.41
$\log_2 8 + \log 6 - \log 81$	$\log_4 8$	$\frac{3}{2}$

(RF4.2, RF4.4, RF4.5)

- In groups of two, students can play *Three in a Row* using expressions that involve exponents and logarithms. Provide each student with a 4 by 4 game board grid. Players take turns placing a marker in a box on the grid. The player must correctly simplify the expression, or solve the problem, and verify with his/her own group that he/she is correct. If the solution is correct, the player places his/her marker in that box. If the solution is incorrect, the other player can steal the box by giving the correct solution. The winner is the player who has three of his/her game markers in a row (vertical, horizontal, or diagonal). A sample grid is shown below.

	1	2	3	4
1	$\log_5 5^2 + \log_2 8$	$\log_2 \sqrt{16}$	$\log_3 9 + \log_6 36$	$4 + 3\log_5 25$
2	$\log_7 49^2$	$5^{x+3} = 5$	$3^{x+1} = 27^{2x+1}$	$2^{2-3x} = \frac{1}{32}$
3	$\log 100 + \log 10$	$2\log 1000 - \log 100^{\frac{1}{2}}$	$2(5)^{3x} = 50$	$3(4)^x = 48$
4	$3^{2x+9} = \frac{1}{81}$	$5^{3+x} = 125^{-2x}$	$3^{\log_4 16}$	$10^{x+3} = \log 100$

(RF5.1, RF4.2, RF4.4, RF4.5)

### Resources/Notes

#### Authorized Resource

*Principles of Mathematics 12*

#### 7.4 Solving Exponential Equations Using Logarithms

SB: pp. 449 - 459

TR: pp. 411 - 419

## Relations and Functions

### Outcomes

*Students will be expected to*

RF5 Continued...

### Achievement Indicators:

RF5.3 *Solve problems that involve the application of exponential equations.*

### Elaborations – Strategies for Learning and Teaching

In the previous unit, students solved problems involving exponential growth and decay, as well as finance applications. These types of problems should now be revisited, as the majority of them involve exponential equations that are solved using logarithms.

When working with the compound interest formula  $A = P(1 + i)^n$ , remind students of the different compounding periods that may occur and that the value of  $i$  is not the annual interest rate but rather the interest rate per compounding period. Ask students to explain the examples such as:

- 5% interest compounded semi-annually means that, in each compounding period, the interest is 2.5%.
- interest compounded semi-annually over 10 years means that the interest is paid 20 times so  $n = 20$ , not 10.

Students should be exposed to situations where it is necessary to determine the initial value, the future value or the number of compounding periods. For example, if a \$1000 deposit is made at a bank that pays 12% per year, compounded annually, students should be able to determine, using logarithms, how long it will take for the investment to reach \$2000.

## General Outcome: Develop algebraic and graphical reasoning through the study of relations.

### Suggested Assessment Strategies

#### *Paper and Pencil*

- Ask students to answer the following:
  - (i) The equation  $A = A_0\left(\frac{1}{2}\right)^{\frac{t}{15}}$  represents the population of a town that halves every 15 years, where  $A$  represents the population remaining after a certain time. In how many years will the population decrease by 25% ?
  - (ii) After taking a cough suppressant, the amount,  $A$ , in mg, remaining in the body is given by  $A = 10(0.85)^t$ , where  $t$  is given in hours.
    - (a) What was the initial amount taken?
    - (b) What percent of the drug leaves the body each hour?
    - (c) How much of the drug is left in the body 6 hours after the dose is administered?
    - (d) How long is it until only 1 mg of the drug remains in the body?
  - (iii) The equation  $A = A_0\left(\frac{1}{2}\right)^{\frac{t}{3}}$  represents the radioactive sample where the half-life is 3 years. If the initial mass of the sample is 67 g, how long will it take for the sample to reach 7 g?
  - (iv) Kelly invests \$5000 with a bank. The value of her investment can be determined using the formula  $y = 5000(1.06)^t$ , where  $y$  is the value of the investment at time  $t$ , in years. Approximately how many years will it take for Kelly's investment to reach a value of \$20 000?
  - (v) \$2000 is invested at 5% per year, compounded semi-annually. How long in months, will it take for the investment to triple in value?

(RF5.2)

### Resources/Notes

#### Authorized Resource

*Principles of Mathematics 12*

#### 7.4 Solving Exponential Equations Using Logarithms

SB: pp. 449 - 459

TR: pp. 411 - 419

## Relations and Functions

### Outcomes

*Students will be expected to*

**RF6 Continued...**

#### Achievement Indicators:

RF6.10 *Graph data, and determine the logarithmic function that best approximates the data.*

RF6.11 *Interpret the graph of a logarithmic function that models a situation, and explain the reasoning.*

RF6.12 *Solve, using technology, a contextual problem that involves data that is best represented by graphs of logarithmic functions and explain the reasoning.*

**LR1 Analyze puzzles and games that involve numerical and logical reasoning, using problem-solving strategies.**

[CN, ME, PS, R]

### Elaborations—Strategies for Learning and Teaching

In the previous units on Polynomials and Exponentials, students worked with linear, polynomial and exponential regressions. They should now use logarithmic regression to model a function of the form

$y = a + b \ln x$  to a set of data. Although they will be given the type of regression to perform on a set of data, ask students why they think a particular model would be a good fit for the particular data set. As students work through problems, ask them to reflect on the following:

- the domain of a logarithmic function is restricted to the set of positive real numbers
- logarithmic regressions are mostly used for phenomena that grow quickly at first and then slow down over time but the growth continues to increase without bound (i.e., the length of cod fish over time)
- exponential regressions are typically used on phenomena where the growth begins slowly and then increases very rapidly as time increases (i.e., bacteria growth)

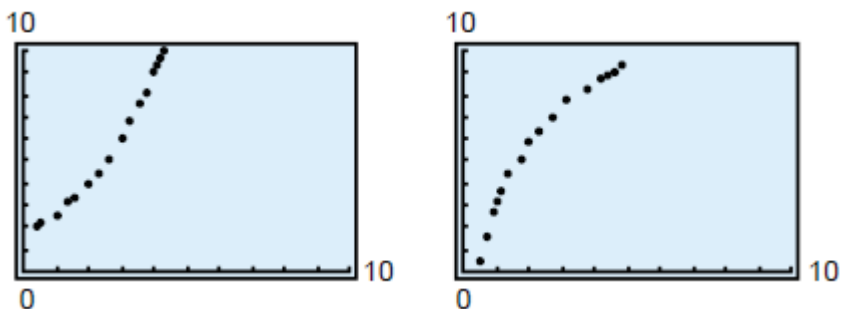
Revisit the puzzles and games, focusing on the strategies students are using. Refer back to pp. 38-47 for additional information.

**General Outcome: Develop algebraic and graphical reasoning through the study of relations.**

### Suggested Assessment Strategies

#### Interview

- Ask students which scatter plot appears to model an exponential function and which models a logarithmic function. Explain your reasoning.



(RF6.11)

#### Paper and Pencil

- Using graphing technology, ask students to determine the equation of the logarithmic regression for the data. They should create a scatter plot and a graph of the model.

$x$	0.5	0.7	0.9	1.0	1.2	1.4	1.8	2.0	2.3	2.7	3.2	3.8
$y$	0.5	1.6	2.7	3.1	3.7	4.4	5.1	5.8	6.4	7.0	7.7	8.3

Ask students to compare the  $y$ -values given by the model with the actual  $y$ -values in the table and explain why this model is or is not a good fit.

(RF6.10, RF6.11, RF6.12)

### Resources/Notes

#### Authorized Resource

*Principles of Mathematics 12*

#### 7.5 Modelling Data Using Logarithmic Functions

SB: pp. 460 - 472

TR: pp. 420 - 428

#### Web Link

The following site compares exponential and logarithmic regressions with specific examples.

<http://emathinstruction.com/sitebuildercontent/sitebuilderfiles/U10L11ExponentialAndLogarithmicRegression.pdf>

#### Games and Puzzles

##### Guess the Number: Binary Search

SB: p.472

TR: p. 427





# Sinusoidal Functions

Suggested Time: 13 Hours

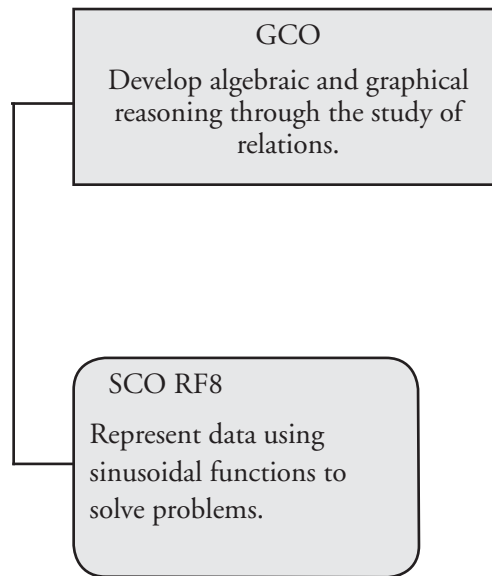
# Unit Overview

## Focus and Context

In this unit, students investigate sinusoidal functions. They first explore the graphs of  $y = \sin x$  and  $y = \cos x$ . These base graphs should be developed in degrees and then radians. Students describe features such as amplitude, period, midline, domain, range, vertical and horizontal shifts, to sketch the graphs of sinusoidal functions and to solve contextual problems.

Using technology, students will create scatter plots for data and use regression to generate modeling functions.

## Outcomes Framework



## Mathematical Processes

[C] Communication  
 [CN] Connections  
 [ME] Mental Mathematics  
 and Estimation

[PS] Problem Solving  
 [R] Reasoning  
 [T] Technology  
 [V] Visualization

## SCO Continuum

Mathematics 1201	Mathematics 2201	Mathematics 3201
Measurement	Geometry	Relations and Functions
M4 Develop and apply the primary trigonometric ratios (sine, cosine, tangent) to solve problems that involve right triangles. [C, CN, PS, R, T, V]	G3 Solve problems that involve the cosine law and the sine law, excluding the ambiguous case. [CN, PS, R]	RF8 Represent data using sinusoidal functions to solve problems. [C, CN, PS, T, V]
Relations and Functions		
not addressed	RF1 Demonstrate an understanding of the characteristics of quadratic functions, including: <ul style="list-style-type: none"> <li>• vertex</li> <li>• intercepts</li> <li>• axis of symmetry</li> </ul> [CN, PS, T, R]	not addressed

## Relations and Functions

### Outcomes

*Students will be expected to*

**RF8 Represent data using sinusoidal functions to solve problems.**

[C, CN, PS, T, V]

### Achievement Indicator:

RF8.1 *Demonstrate an understanding of angles expressed in degrees and radians.*

### Elaborations—Strategies for Learning and Teaching

In Mathematics 1201, students used the primary trigonometric ratios and the Pythagorean theorem to solve right triangle problems (M4).

In Mathematics 2201, they solved problems that involved the cosine law and the sine law, excluding the ambiguous case (G3). In this unit, students describe the characteristics of sinusoidal functions by analyzing the graph and its corresponding equation:  $y = a \sin b(x - c) + d$  and  $y = a \cos b(x - c) + d$ .

They should also determine the equation of the sinusoidal regression function that models a set of data.

Ask students to think about the different ways they can measure things. Length can be measured in centimeters or in yards, temperature can be measured in degrees Celsius or Fahrenheit. Ask students to think about the different ways to measure angles. They may refer to degree measure or use the concept of turns. Introduce students to radian measure as an alternative way to express the size of an angle. This will be their first time being exposed to the concept of radian measure.

Ask students to visualize a circle where the radius equals one unit. Teachers can use the following questions to help students connect the concepts of degrees and radians:

- What is the circumference of a circle?
- What is a complete revolution of a circle?
- Why must the two equations be equal to each other?

Students should be able to state that  $2\pi$  radians =  $360^\circ$ . Ask them to solve this equation for 1 radian, resulting in  $1 \text{ radian} \approx 57.3^\circ$ .

They should then manipulate the relationship further to get a sense for the different angles. If  $2\pi = 360^\circ$ , for example, then  $\pi = 180^\circ$ ,  $\frac{\pi}{2} = \frac{1}{2}\pi = 90^\circ$ ,  $\frac{\pi}{4} = \frac{1}{4}\pi = 45^\circ$ ,  $\frac{\pi}{6} = \frac{1}{6}\pi = 30^\circ$  and  $\frac{\pi}{3} = \frac{1}{3}\pi = 60^\circ$ .

Students should be able to change a degree measure to radian measure and each radian measure to degree measure. Ask students to determine, for example, an approximate radian measure for  $160^\circ$ . They could write and solve a proportion. Using the fact that  $180^\circ = \pi$  radians, ask students to let  $x$  be the radian measure of  $160^\circ$ . Ask students to solve the proportion  $\frac{180^\circ}{\pi} = \frac{160^\circ}{x}$  for  $x$ . Similarly, if students converted 0.7 radians to degrees, they solve the following proportion  $\frac{180^\circ}{\pi} = \frac{x}{0.7}$ . Answers may be given in exact or approximate measure.

Alternatively, students could use the relationship  $\pi$  radians =  $180^\circ$  to solve for 1 radian and 1 degree. Dividing both sides of the equation by  $\pi$  results in  $1 \text{ radian} = \frac{180^\circ}{\pi}$ . Dividing both sides of the equation by  $180^\circ$  results in  $1^\circ = \frac{\pi}{180^\circ}$ .

## General Outcome: Develop algebraic and graphical reasoning through the study of relations.

### Suggested Assessment Strategies

#### *Interview*

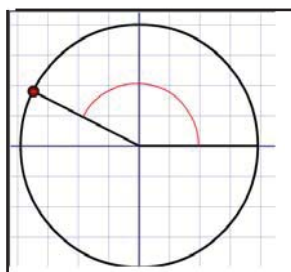
- Ask students to provide their opinion on the advantages and disadvantages of measuring in degrees versus measuring in radians. (RF8.1)

#### *Paper and Pencil*

- Ask students to approximate the value for each conversion:
  - 0.4 radians to degrees
  - 4.5 radians to degrees
  - $150^\circ$  to radians
  - $470^\circ$  to radians
 (RF8.1)

#### *Performance*

- Ask students to draw an angle such as the one shown below. They should then give their drawing to a classmate and ask their partner to estimate the measure in radians. After the activity, ask the class to discuss the various strategies that can be used.



(RF8.1)

### Resources/Notes

#### Authorized Resource

*Principles of Mathematics 12*

#### 8.1 Understanding Angles

Student Book (SB): pp. 484 - 490  
 Teacher Resource (TR): pp. 456-461

#### Web Link

This interactive site allows students to see the relationship between degrees and radians for a unit circle.

<http://www.mathlearning.net/dl2004/Demos/unitCircle.html>

## Relations and Functions

### Outcomes

*Students will be expected to*

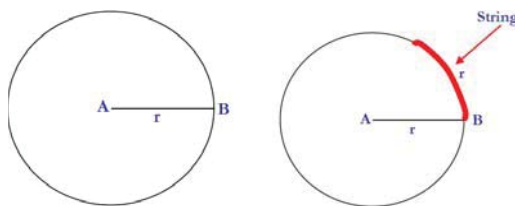
RF8 Continued...

**Achievement Indicator:**

RF8.1 *Continued*

### Elaborations—Strategies for Learning and Teaching

The following activity could be used as an alternative method for developing the concept of a radian. Present the class with a visual of a large circle. This could be one within the classroom or a circle on the gymnasium floor. Use a piece of rope (or string) to represent the radius ( $r$ ) of the circle. A radius is a straight line segment and the arc length is curved because it is part of the circumference but they have the same length. Point out to students that one radian is the angle made by taking the radius and wrapping it along the edge of the circle. Ask them to predict how many pieces of rope, of this length, it would take to represent the circumference of the circle.



Students should recognize that the circumference is a little more than 6 radius lengths (approximately  $6\frac{1}{4}$  pieces of rope (of length  $r$ ) are needed for one complete circle). They can, therefore, estimate that 1 radian  $\approx 60^\circ$ . Discuss with students if the size of the radius of a circle has an effect on the size of 1 radian.

It would be appropriate to discuss the reasons radian measure is used with sinusoidal functions. The advantage of radians is that it is directly related to the radius of the circle (which is also the amplitude). This means that the units on the  $x$  and  $y$  axis are consistent and the graph of the sine curve will have its true shape, without vertical exaggeration. If degrees and length are used for the  $x$  and  $y$  axes respectively, then the shape of the two graphs may change depending on the scale.

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**General Outcome: Develop algebraic and graphical reasoning through the study of relations.**

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**Suggested Assessment Strategies**

*Observation*

- A student approximated  $45^\circ$  to be 0.8 in radian measure. Ask students to use this measure to estimate the missing values.
  - (i)  $90^\circ = ?$  radians
  - (ii)  $?^\circ = 4$  radians
  - (iii)  $135^\circ = ?$  radians
  - (iv)  $?^\circ = 3.2$  radians

(RF8.1)

**Resources/Notes**

**Authorized Resource**

*Principles of Mathematics 12*

**8.1 Exploring the Characteristics of Exponential Functions**

SB: pp. 334 - 337

TR: pp. 326 - 329

## Relations and Functions

### Outcomes

Students will be expected to

RF8 Continued...

#### Achievement Indicator:

RF8.2 Describe, orally and in written form, the characteristics of a sinusoidal function by analyzing its graph.

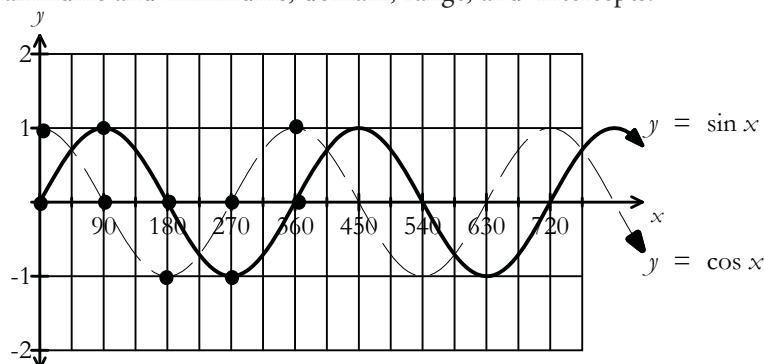
### Elaborations—Strategies for Learning and Teaching

Examples of a repeated cycle or a pattern exists in a variety of everyday items such as wallpaper, fabric prints, flooring and computer graphic designs. When working with functions, if a pattern repeats regularly over some interval of the domain, then the function is periodic. A ferris wheel demonstrates periodic behaviour since it completes one rotation every  $t$  minutes.

Students should explore the patterns in the sine and cosine ratios. They could use graphing technology and a table to record the points. Remind students to choose an appropriate increment for  $\theta$  when graphing  $y = \sin\theta$  and  $y = \cos\theta$  from  $0^\circ$  to  $720^\circ$ . Similarly, repeat this activity in radian measure for the domain  $0$  to  $4\pi$ .

Introduce students to the terms period, amplitude and midline (i.e., sinusoidal axis), and their definitions as they create the graphs of  $y = \sin\theta$  and  $y = \cos\theta$ . This should then lead into a discussion around sinusoidal function.

Draw attention to the 5 key points associated with each graph, since these points help us determine the characteristics of the graph. In addition to the characteristics listed, students should also identify local maximums and minimums, domain, range, and intercepts.



For cosine, students should observe that one complete wave can be seen from the maximum point  $(0, 1)$  to the next maximum point at  $(360^\circ, 1)$ . For sine, one complete wave can be seen from an  $x$ -intercept at  $(0, 0)$  to the  $x$ -intercept at  $(360^\circ, 0)$ . Ask students if there are other points that can be used to show a complete wave for either the sine or cosine function. Students should conclude that both curves have a period of  $360^\circ$  (or  $2\pi$  radians), amplitude of 1, and a midline defined by the equation  $y = 0$ .

Using the graphs as a visual aid, students should observe the horizontal shift between the two functions. They should notice that the graph of  $y = \cos\theta$  is related to the graph of  $y = \sin\theta$  by a shift of  $90^\circ$  ( $\frac{\pi}{2}$ ) to the left.



## General Outcome: Develop algebraic and graphical reasoning through the study of relations.

### Suggested Assessment Strategies

#### *Interview*

- Using technology, ask students to create the graphs of  $y = \sin \theta$  and  $y = \cos \theta$ . Use the following questions to guide discussion:
  - (i) What is the maximum value of  $\sin \theta$  and  $\cos \theta$ ? the minimum value?
  - (ii) For what values of  $\theta$  is there a maximum value? a minimum value?
  - (iii) What are the  $x$ -intercepts and  $y$ -intercept?
  - (iv) For what values of  $\theta$  is the function positive? For what values is it negative?
  - (v) Do the graphs demonstrate periodic behaviour? Explain your reasoning.
  - (vi) What is the period? What interval did you use to determine the value of the period?
  - (vii) What is the height or amplitude of the graph?
  - (viii) Where is the midline?
  - (ix) What comparisons can you make between the sine and cosine function?

(RF8.2)

#### *Observation*

- Create a model of the coordinate grid on the classroom floor. Use a metre stick, that can be moved physically, to represent the radius of the circle. Ask students to record and graph the distance, in metres, from the tip of the metre stick to the  $x$ -axis over  $360^\circ$ . These values will produce the sine curve. Ask students to repeat the process measuring the distance, in metres, to the  $y$ -axis over  $360^\circ$ . These values produce the cosine curve.

(RF8.2)

### Resources/Notes

#### Authorized Resource

*Principles of Mathematics 12*

#### 8.2 Exploring Graphs of Periodic Functions

SB: pp. 491 - 496

TR: pp. 462 - 467

#### Web Link

The following site contains an applet that draws the sine and cosine function

[http://www.walter-fendt.de/m14e/sincostan\\_e.htm](http://www.walter-fendt.de/m14e/sincostan_e.htm)

## Relations and Functions

### Outcomes

Students will be expected to

#### RF8 Continued... Achievement Indicators:

RF8.3 Interpret the graph of a sinusoidal function that models a situation, and explain the reasoning.

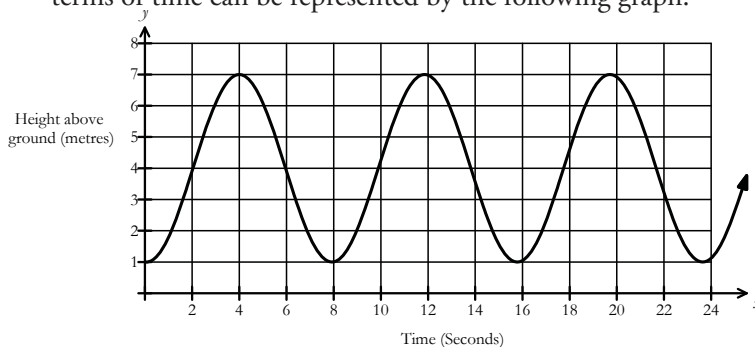
RF8.4 Describe, orally and in written form, the characteristics of a sinusoidal function by analyzing its equation.

### Elaborations – Strategies for Learning and Teaching

Provide students with a variety of graphs where some graphs are periodic but not sinusoidal and others are periodic and sinusoidal. Comparing the graphs, students should be able to conclude that all sinusoidal functions are periodic but not all periodic functions are sinusoidal.

Sinusoidal functions can be used as models to solve problems that involve repeating or periodic behaviour. Provide students with various graphs that model a situation and ask them to identify the characteristics of a sinusoidal function and relate its significance within the context of the problem. Teachers could use the following example:

- While riding on a Ferris wheel, Mason's height above the ground in terms of time can be represented by the following graph.



Ask students to identify the period, range, amplitude, intercepts, and the equation of the midline. They should explain what the values represent within the context of this problem.

In Mathematics 2201, students worked with the quadratic function  $y = a(x - h)^2 + k$ , and manipulated the parameters  $a$ ,  $h$ , and  $k$  to see how they affected the graph (RF1). Students will now investigate how the graph is affected by the values of  $a$ ,  $b$ ,  $c$  and  $d$  in sinusoidal functions written in the form  $y = asinb(x - c) + d$  and  $y = acosb(x - c) + d$ . It would be beneficial if students investigated the parameters separately so they can describe how each one affects the graph. For the purpose of this activity, students should be familiar with working with radians or degrees.

Using graphing technology, students should examine the effects of manipulating the value of  $a$  by comparing the sinusoidal functions  $y = \sin x$  and  $y = a\sin x$ , where  $a > 0$ . For example, as they compare the graphs of  $y = 2\sin x$ ,  $y = 5\sin x$  and  $y = 0.5\sin x$  to the graph of  $y = \sin x$ , use the following questions to promote student discussion:

- What happens to the amplitude if  $a > 0$  ?
- Is the shape of the graph affected by the parameter  $a$ ?
- How is the range affected by the parameter  $a$ ?
- Will the value of  $a$  affect the cosine graph in the same way that it affects the sine graph? Why or why not?

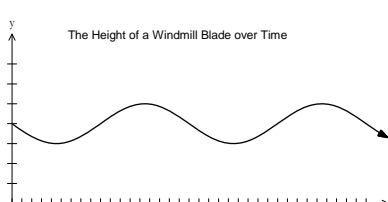
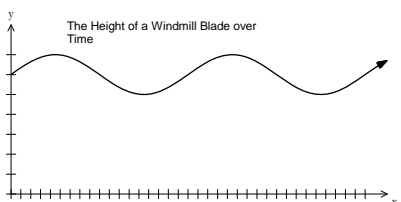
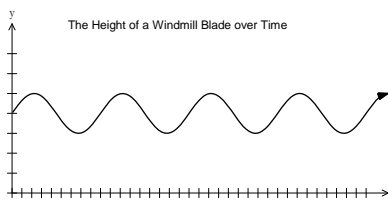
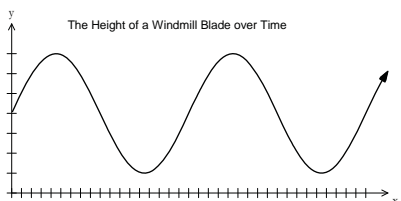
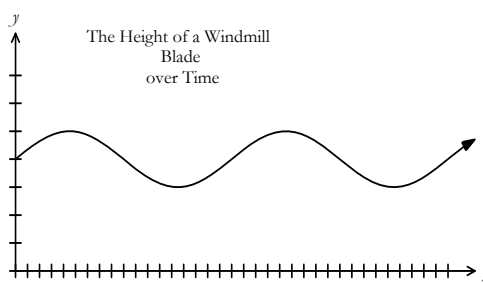
**General Outcome: Develop algebraic and graphical reasoning through the study of relations.**

**Suggested Assessment Strategies**

*Interview*

- A company is experimenting with a new type of windmill. The graph below shows the path of a blade on the original windmill over time. Ask students to describe what has changed and what has stayed the same in the new models. (Note: The scale is the same on all graphs)

ORIGINAL MODEL



**Resources/Notes**

**Authorized Resource**

*Principles of Mathematics 12*

**8.3 The Graphs of Sinusoidal Functions**

SB: pp. 497 - 512

TR: pp. 468 - 476

**8.4 The Equations of Sinusoidal Functions**

SB: pp. 516 - 532

TR: pp. 477 - 486

(RF8.3)

## Relations and Functions

### Outcomes

*Students will be expected to*

RF8 Continued...

### Achievement Indicator:

RF8.4 *Continued*

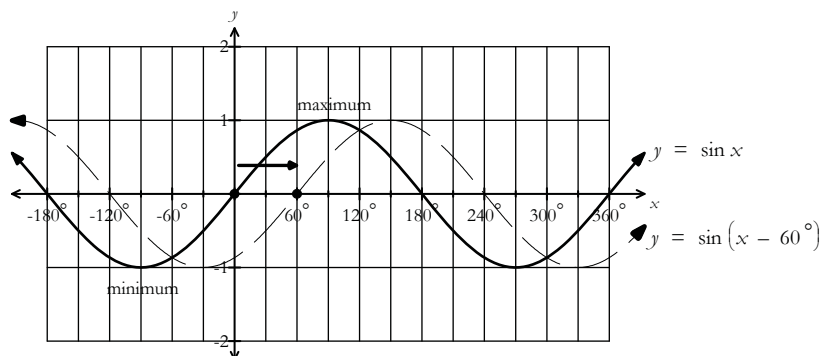
### Elaborations—Strategies for Learning and Teaching

Similarly, students should examine the value of  $d$  by comparing the sinusoidal function  $y = \sin x$  and  $y = \sin(x) + d$ . Ask them to compare the graphs of  $y = \sin(x) + 2$ ,  $y = \sin(x) - 3$  to the graph of  $y = \sin x$  and answer the following questions:

- How does each graph change when compared to  $y = \sin x$  ?
- How is the value of  $d$  related to the equation of the midline?
- Is the shape of the graph or the location of the graph affected by the parameter  $d$  ?
- Is the period affected by changing the value of  $d$  ?
- Will the value of  $d$  affect the cosine graph in the same way that it affects the sine graph? Why or why not?

Students should manipulate parameter  $b$  by comparing the sinusoidal functions  $y = \sin x$  and  $y = \sin bx$ , where  $b > 0$ . They could describe how the graph of  $y = \sin 2x$  and  $y = \sin 0.5x$  compares to the graph of  $y = \sin x$ . Ask students what effect does varying  $b$  have on the period. They should notice the period of  $y = \sin 2x$  is  $180^\circ$  while the period of  $y = \sin 0.5x$  is  $720^\circ$ . Students may need guidance to make the connection that the period of a sinusoidal function is  $\frac{360^\circ}{b}$  (or  $\frac{2\pi}{b}$ ). Provide students with an opportunity to graph the cosine function in a similar way so they can conclude that parameter  $b$  affects both functions in the same way.

When manipulating the value of  $c$ , students should identify key points located on the sine and cosine graph. They are aware that the graph of the cosine and sine function have different starting points. In order to determine the horizontal shift (or phase shift), they have to observe if the key point has shifted left or right. Ask students to first compare the functions  $y = \sin x$  to  $y = \sin(x - c)$ . They could describe how the graph of  $y = \sin(x - 60^\circ)$  and  $y = \sin(x + 30^\circ)$  compares to the graph of  $y = \sin x$ . One of the key points of  $y = \sin x$  is the point  $(0, 0)$ . This point intersects the midline at  $x = 0$  going from a minimum to a maximum. The graph of  $y = \sin(x - 60^\circ)$ , for example, intersects the midline at  $x = 60^\circ$  resulting in a horizontal shift of  $60^\circ$  to the right.

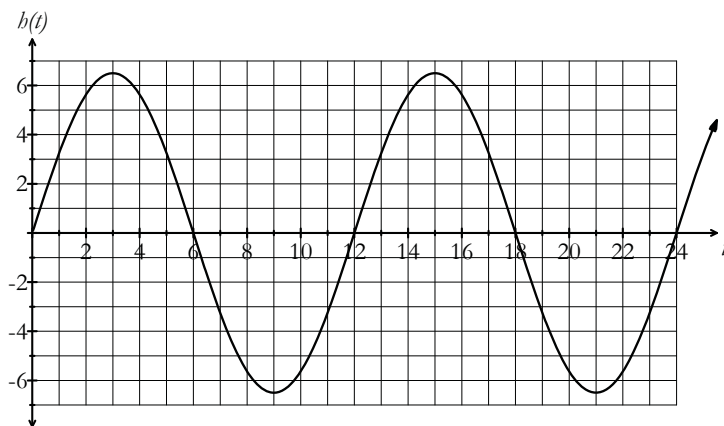


**General Outcome: Develop algebraic and graphical reasoning through the study of relations.**

### Suggested Assessment Strategies

#### *Paper and Pencil*

- The following graph represents the rise and fall of sea level in part of the Bay of Fundy, where  $t$  is the time, in hours, and  $h(t)$  represents the height relative to the mean sea level:



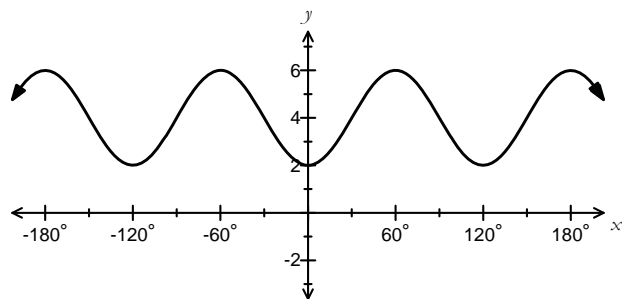
Ask students the following questions:

- What is the range of the tide levels?
- What does the equation of the midline represent in the graph?
- What is the period of the graph?
- The equation of the sinusoidal function is represented by  $h(t) = 6.5 \sin \frac{\pi}{6} t$ . Calculate the period from the equation and compare it to your answer in (iii).

(RF8.3, RF8.4)

#### *Journal*

- The equation of the graph below in  $y = a \sin b(x - c) + d$  form is  $y = 2 \sin 3(x - 30^\circ) + 4$ . If you were not given this equation, explain how you would find the values of  $a$ ,  $b$ ,  $c$  and  $d$ . Which value(s) did you find first? Why? Which value was the most difficult to determine? Which values would change if this is a transformed cosine curve?



(RF8.4)

### Resources/Notes

#### Authorized Resource

*Principles of Mathematics 12*

8.4 The Equations of Sinusoidal Functions

SB: pp: 516-532

TR: pp: 477-486

## Relations and Functions

### Outcomes

*Students will be expected to*

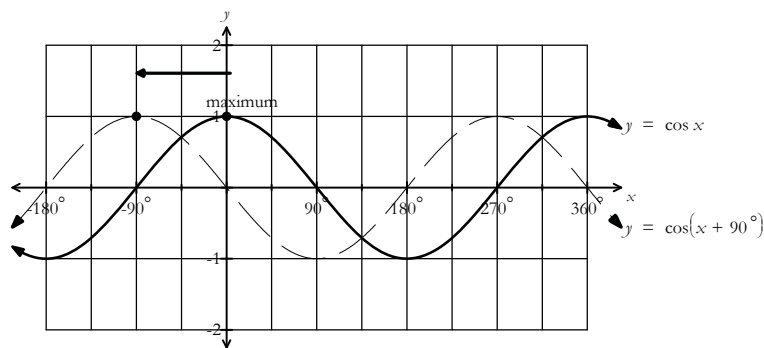
RF8 Continued...

### Achievement Indicators:

RF8.4 *Continued*

### Elaborations—Strategies for Learning and Teaching

When determining the horizontal shift for the cosine function, one of the key points students can consider is the point  $(0, 1)$ , which is a maximum point on the graph. Ask students to compare the functions  $y = \cos x$  to  $y = \cos(x - c)$ . They could describe how the graph of  $y = \cos(x + 90^\circ)$  and  $y = \cos(x - 30^\circ)$  compares to the graph of  $y = \cos x$ . The graph of  $y = \cos(x + 90^\circ)$ , for example, has a maximum point at  $(-90^\circ, 1)$  which results from a horizontal shift of  $90^\circ$  to the left.



Once students have been exposed to all the different parameters, provide them with a variety of equations where they have to state the amplitude, equation of the midline, range, period as well as the horizontal translation. Encourage them to confirm their description by sketching the graph using graphing technology.

Students sometimes do not recognize the difference parentheses make in the meaning of horizontal and vertical translations. When comparing  $y = \cos(x + 2)$  to  $y = \cos x + 2$ , they should observe that  $y = \cos(x + 2)$  results in a horizontal translation while  $y = \cos x + 2$  results in a vertical translation. It may be beneficial if students write  $y = \cos x + 2$  as  $y = \cos(x) + 2$ . This would reinforce their understanding that  $y = \cos(x) + 2$  does not result in a horizontal translation (i.e.,  $y = \cos(x + 0) + 2$ ) but rather a vertical translation of 2 units upward.

RF8.5 *Match equations in a given set to their corresponding graphs.*

Students should match a sinusoidal graph with its corresponding equation. Using the equations given, they will work through the process of elimination until they reach one that satisfies the given graph. Encourage students to explain their reasoning as to why they are eliminating choices as they work through the solution.

## General Outcome: Develop algebraic and graphical reasoning through the study of relations.

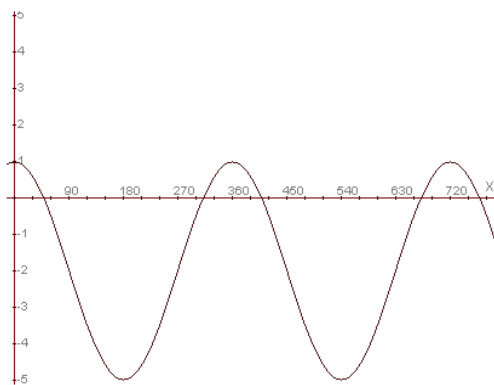
### Suggested Assessment Strategies

#### *Paper and Pencil*

- The temperature of an air-conditioned home on a hot day can be modelled using the function  $t(x) = 1.5(\cos 15^\circ x) + 20$ , where  $x$  is the time in minutes after the air conditioner turns on and  $t(x)$  is the temperature in degrees Celsius. Ask students to answer the following:
  - What are the maximum and minimum temperatures in the home?
  - What is the temperature 10 minutes after the air conditioner has been turned on?
  - What is the period of the function? How would you interpret this value in this context?

(RF8.4)
- The depth of water  $d(t)$ , in metres, in a seaport can be approximated by the function  $d(t) = 2.5\sin 0.164\pi(t - 1.5) + 13.4$ , where  $t$  is the time in hours. Ask students the following:
  - Graph the data.
  - What is the period of the tide? What information does the period give you?
  - A cruise ship needs a depth of at least 12m of water to dock safely. For how many hours per tide cycle can the ship dock safely?

(RF8.4)
- Ashley created the following graph for the equation  $y = 3\sin(x - 90^\circ) + 2$  as shown below.



Ask students to identify the error Ashley made. They should then construct the correct graph.

(RF8.4, RF8.5)

### Resources/Notes

#### Authorized Resource

*Principles of Mathematics 12*

#### 8.4 The Equations of Sinusoidal Functions

SB: pp: 516-532

TR: pp: 477-486

## Relations and Functions

### Outcomes

*Students will be expected to*

RF8 Continued...

#### Achievement Indicators:

RF8.6 *Graph data, and determine the sinusoidal function that best approximates the data.*

RF8.7 *Solve, using technology, a contextual problem that involves data that is best represented by graphs of sinusoidal functions, and explain the reasoning.*

### Elaborations – Strategies for Learning and Teaching

Students have been previously exposed to finding a polynomial, exponential and logarithmic regression equation. They should now progress to a sinusoidal regression.

In this unit, students have worked with the sine function in terms of amplitude, period, horizontal and vertical translation. Examples such as sound waves and height of tides can be modelled with sinusoidal functions. Ask students to create a scatterplot of the data and explain why a sinusoidal function can be used to model the data. They should then determine and graph the sinusoidal regression function. Teachers should inform students when a sinusoidal regression is performed, the calculator generates an equation  $y = a \sin(bx + c) + d$  in radian measure. Remind students to check their graph to make sure it is a reasonable fit to the data. This reinforces the concept that regression equations are best-fit equations of available data, not perfect models.

Regression analysis can be performed using a variety of technologies, such as graphing calculators and their emulators and graphing applications available for smartphones, tablets or computers.

Students should analyze the graph to determine the maximum and minimum values, the amplitude, and the period. They should also interpolate or extrapolate values that can be predicted by reading values from the graph. Students can also use the equation of the sinusoidal regression function to solve for values of the independent variable (i.e., given  $x$  evaluate for  $y$ ).

Students will not be expected to identify the value of the horizontal translation  $c$  using the regression equation  $y = a \sin(bx + c) + d$ . They would have to factor out the  $b$  value which is a concept beyond the scope of the outcome.

This would be a good opportunity to ask students to investigate environmental data that is periodic in nature and present a scatterplot and sinusoidal graph of best fit for the data. They should use the data to interpolate and extrapolate during the presentation. Students should extend their findings to make other generalizations about that environment.



## General Outcome: Develop algebraic and graphical reasoning through the study of relations.

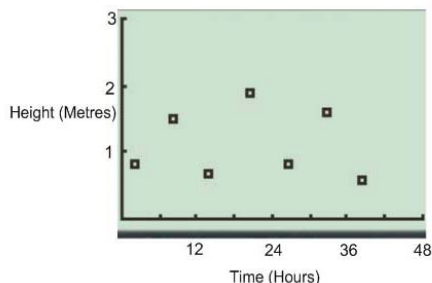
### Suggested Assessment Strategies

#### *Paper and Pencil*

- A marine biologist recorded the tide times in the Greater St. Lawrence area for July 17<sup>th</sup> and 18<sup>th</sup>, 2012 in the chart below.

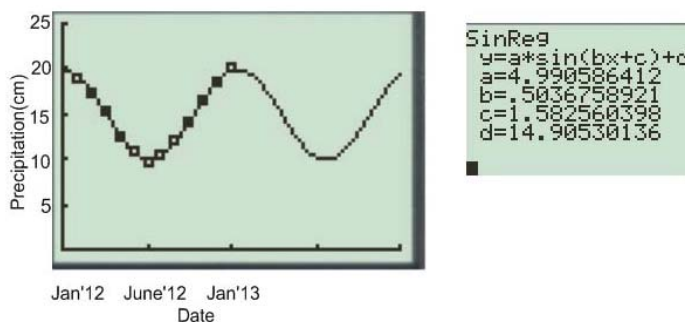
Time (Hour:Minutes)	Height (Metres)
01:50	0.8
07:55	1.5
13:35	0.7
20:38	1.9
26:25	0.8
32:40	1.6
38:15	0.6

He recorded the data in his TI-83 calculator below:



Ask students to use a curve of best fit to estimate the water height after 15 hours of observation. Can this model be used to predict the tide height over the next 2 days? The next week? The next month? Discuss with your classmates. (RF8.6, RF8.7)

- Meteorologist Bryan Nodden recorded the average precipitation in Grand Falls-Windsor for 2012 and created a sinusoidal regression for the data. Ask students to use the graph to predict the amount of precipitation in August 2013. They should use the equation to predict the amount of precipitation in March 2017. Ask them to consider what some problems might be with the use of a sinusoidal regression to predict future precipitation.



(RF8.6, RF8.7)

### Resources/Notes

#### Authorized Resource

*Principles of Mathematics 12*

#### 8.5 Modelling Data with Sinusoidal Functions

SB: pp: 533-548

TR: pp: 487-494

#### Web Link

<http://tides.gc.ca>



# Financial Mathematics: Borrowing Money

Suggested Time: 11 Hours

# Unit Overview

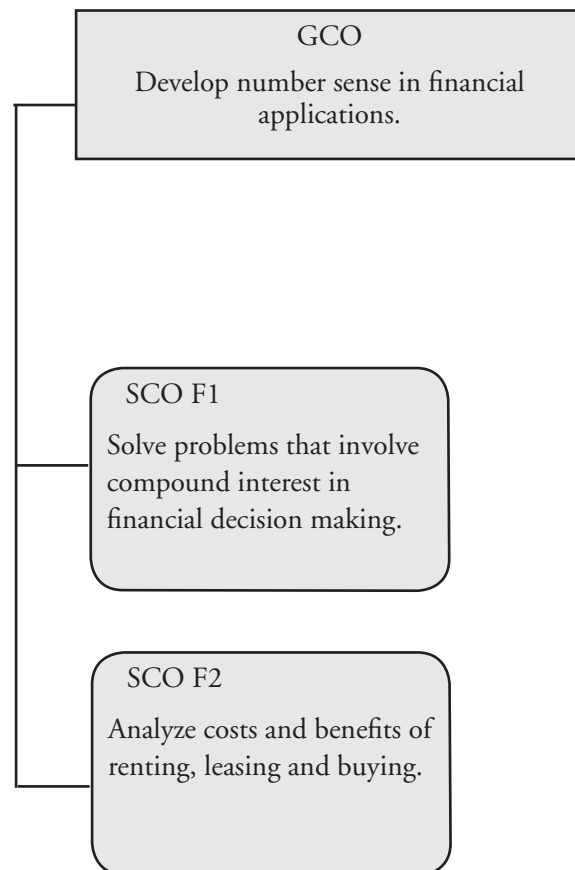
## Focus and Context

In today's complex financial world, being financially literate is a critical life skill. It is important to include financial education within the teaching of mathematics.

Developing the set of skills and knowledge that allows students to make appropriate decisions when managing personal finances is important. Students should consider topics such as the following:

- financial concepts such as compound interest
- costs and benefits of renting, leasing, and buying

## Outcomes Framework



Mathematical Processes

- [C] Communication
- [CN] Connections
- [ME] Mental Mathematics and Estimation
- [PS] Problem Solving
- [R] Reasoning
- [T] Technology
- [V] Visualization

SCO Continuum

Mathematics 1201	Mathematics 2201	Mathematics 3201
Financial Mathematics		
not addressed	not addressed	<p>F1 Solve problems that involve compound interest in financial decision making. [C, CN, PS, T, V]</p> <p>F2 Analyze costs and benefits of renting, leasing and buying. [CN, PS, R, T]</p>

## Financial Mathematics

### Outcomes

*Students will be expected to*

**F1 Solve problems that involve compound interest in financial decision making.**

[C, CN, PS, T, V]

### Achievement Indicator:

F1.1 *Explain the advantages and disadvantages of compound interest and simple interest.*

F1.2 *Identify situations that involve compound interest.*

F1.3 *Solve a contextual problem that involves compound interest.*

### Elaborations – Strategies for Learning and Teaching

While compound interest is beneficial to the person receiving the interest, it costs the person paying the interest on a loan or credit card a lot of money, as interest is charged on interest.

In this unit, students will solve problems that involve single payment loans and regular payment loans. The use of technology such as spreadsheets or a financial application is important here.

Students have been exposed to both simple and compound interest in the unit on Exponential Functions (RF5). Ask students for examples where simple interest is used, as well as those that involve compound interest. Personal loans from family members, for example, may use simple interest. Ask students to consider other possible advantages of borrowing money from family. In some cases family members may offer the money at a very low interest rate or no interest rate at all. Simple interest is also normally used for loans or investments of a year or less.

Many products offered by financial institutions, however, typically charge compound interest. Ask students for their input and subdivide their responses into borrowing options and investments.

Borrowing	Investing
Loan	Savings Account
Credit Card	Chequing Account
Mortgage	GIC
Line of Credit	Canada Savings Bond
Student Loan	

Discuss with students that GICs and Canada Savings Bonds can have either simple or compound interest.

When discussing the advantages and disadvantages of simple versus compound interest, ask students to explore options that are available to the customer. Ask students if the perspective of the lender and the perspective of the customer be considered. Ask students to comment on the following points:

- An advantage for the lender would typically be a disadvantage for the customer.
- Depending on whether one is investing or borrowing, determines whether simple interest or compound interest is more advantageous.

Remind students of the formulas used for each type of interest:

- simple interest  $A = P(1 + rt)$
- compound interest  $A = P(1 + i)^n$

## General Outcome: Develop number sense in financial applications.

### Suggested Assessment Strategies

#### *Paper and Pencil*

- Suppose Peter borrows \$1000 from his parents. Ask students to answer the following:
  - (i) How much will he have to pay back in 2 years if they charge 3% interest per year?
  - (ii) How much will you have to pay back if they charge 3% interest compounded monthly?
  - (iii) Explain which is the better option for him.

(F1.1, F1.2, F1.3)
  
- As a pre-assessment, ask students to complete a snowball activity. Provide all students with a blank sheet of paper. Ask the following question to the class “What is interest (on a loan, etc.)?” Give the students a few minutes to write on their sheet what they think interest is. Ask the students to toss the "snowballs" to the front of the class and then collect one snowball each. They should take turns to read aloud what is written on the snowball they have chosen. The teacher should then summarize their ideas and generate a discussion on the topic to start the unit.
 

(F1.1, F1.2)
  
- Ask students to predict which of the following investments would yield the greater return:
  - Option 1: \$1000 at 3.5% annual simple interest
  - Option 2: \$1000 at 3% annual compound interest

They should verify their prediction by calculating the value of their investments after 5 years, 10 years and 15 years.

(F1.1, F1.2, F1.3)

#### *Journal*

- Peter and Jennifer both say that their savings accounts earn 2% interest. However, after one year Peter’s \$1000 investment has grown to \$1020.00 whereas Jennifer’s investment has grown to \$1020.18. Ask students to explain the difference.
 

(F1.1, F1.2, F1.6)

### Resources/Notes

#### Authorized Resource

*Principles of Mathematics 12*

#### 10.1 Analyzing Loans

Student Book (SB): pp: 636-653  
 Teacher Resource (TR): pp.592-604

#### Web Links

A compound interest calculator can be found at:

<http://math.about.com/library/blcompoundinterest.htm>

An interactive activity involving Texas Instruments can be found at: [www.education.ti.com/calculators/downloads/US/Activities/Detail?id=8015](http://www.education.ti.com/calculators/downloads/US/Activities/Detail?id=8015)

The following site provides an online simple interest calculator: [www.pine-grove.com/online-calculators/simple-interest-calculator.htm](http://www.pine-grove.com/online-calculators/simple-interest-calculator.htm)

## Financial Mathematics

### Outcomes

*Students will be expected to*

F1 Continued...

#### Achievement Indicators:

F1.1, F1.2, F1.3 *Continued*

F1.4 *Compare, in a given situation, the total interest paid or earned for different compounding periods.*

F1.5 *Determine the total interest of a loan given the principal, interest rate and number of compounding periods.*

### Elaborations – Strategies for Learning and Teaching

Students should solve problems where different options are available and explain why they chose a specific option. Ask students to think about a scenerio such as:

- James intends to go to university in five years. His grandmother decides to invest \$2000 in a Guaranteed Investment Certificate (GIC) to help with his first-year expenses.

Ask students to answer the following questions:

- How much would the GIC be worth in 5 years if she chooses a simple interest GIC at 3% annual interest?
- How much would it be worth if the interest is compounded monthly?
- Which option is better for the bank?
- Which is better for James?

Remind students that simple interest increases linearly, whereas compound interest increases exponentially. Also, remind them that 3% interest with "monthly compounding" does not mean 3% per month. It means 0.25% per month.

In the unit on Exponential Functions, students solved problems where the interest rate and compounding period varied. They are familiar with compounding periods such as daily, monthly, quarterly, semi-annually and annually.

To be successful, students should be able to distinguish between biweekly payments, accelerated biweekly payments and semi-monthly payments. Teachers could ask students to consider a loan payment of \$600 per month and work through the payments to help them identify the differences.

- Biweekly  $\rightarrow 600 \times 12 \div 26 = \$276.92$   
paid 26 times per year
- Semi Monthly  $\rightarrow 600 \div 2 = \$300.00$ ,  
paid 24 times per year (1<sup>st</sup> and 15<sup>th</sup> of each month)
- Accelerated Biweekly  $\rightarrow 600 \div 2 = \$300.00$   
paid 26 times per year



**General Outcome: Develop number sense in financial applications.**

**Suggested Assessment Strategies**

*Performance*

- Ask students to create a word wall to display the different terminology in this unit. They could create a poster that shows their word and a brief definition and/or illustration of it. Below is a list of suggested words to use.

Biweekly	Leasing	
Monthly	Financing	Simple Interest
Semi monthly	Mortgage	Compound Interest
Annual	Future Value	Borrowing
Maturity Date	Appreciation	Investing
Amoritization	Depreciation	Balance
Principal	Lender	Loan
Line of credit	Investor	

(F1.1, F1.2)

*Paper and Pencil*

- Ask students to complete the following spreadsheet.

	A	B	C	D	E	F	G	H
1	Principal	Rate per annum %	Compound	Compound periods in one year	Rate per compound period	Term (years)	# of compound periods	Amount
2	\$1000	12	annually	1	12	2	2	\$1254.40
3			semi-annually					
4			monthly					
5			biweekly					

Ask them which compounding period provides the most return on their investment?

(F1.4)

**Resources/Notes**

**Authorized Resource**

*Principles of Mathematics 12*

**10.1 Analyzing Loans**

SB: pp: 636-653

TR: pp.592-604

## Financial Mathematics

### Outcomes

*Students will be expected to*

F1 Continued...

### Achievement Indicator:

F1.4, F1.5 *Continued*

### Elaborations—Strategies for Learning and Teaching

A loan can involve regular loan payments over the term of the loan or a single payment at the end of the term. Students should first work through examples where a loan is paid using a single payment at the end of the term. They should also compare the total interest paid for different compounding periods given the same initial principal, interest rate and term. When making financial decisions, it is important for students to understand the rate of interest charged, as well as the compounding, as these can create large differences over long periods of time. Ask students to solve a problem similar to the following:

- Mary borrows \$1000 at 10% interest, compounded semi-annually. Sean borrows \$1000 at 10% interest compounded annually. How much interest will each pay at the end of two years?

Students should notice interest accumulates faster when there is an increase in the frequency of compounding. Encourage students to work through the term of the loan so they have an opportunity to describe and explain how the interest is calculated. They can then progress to using the compound interest formula,  $A = P(1 + i)^n$  to determine the value of the loan and use  $I = A - P$  to determine the total interest.

Mary: charged \$50 interest at the end of 6 months increasing the principal to \$1050. At the end of the first year, the interest charged is \$52.50 (5% of \$1050) resulting in a principal of \$1102.50. As students continue this for each compounding period, at the end of two years, the principal amount is now \$1215.51 (i.e.,  $A = 1000(1 + \frac{0.10}{2})^{2(2)}$ ). The total interest of the loan results in \$215.51.

Sean: charged \$100 interest at the end of the first year increasing the principal to \$1100. At the end of the two years, the principal amount is now \$1210 (i.e.,  $A = 1000(1+0.10)^2$ ). The total interest of the loan results in \$210.

Teachers should provide students examples of a single loan payment at the end of the loan's maturity date.

- a farmer making a single lump sum payment on his loan after his crop has been harvested
- a payday loan offered by certain financial service providers.

**General Outcome: Develop number sense in financial applications.****Suggested Assessment Strategies***Paper and Pencil*

- Marie takes out a loan for \$5000 at 4% compounded annually. Ask students to answer the following questions:
  - (i) How much total interest will she pay if the loan is for 6 years?
  - (ii) If the interest is compounded monthly, how much total interest will be paid?
  - (iii) What if the interest is compounded daily?
  - (iv) What happens to the amount of interest as the frequency of compounding changes?

(F1.4)

**Resources/Notes****Authorized Resource***Principles of Mathematics 12***10.1 Analyzing Loans**

SB: pp: 636-653

TR: pp.592-604

## Financial Mathematics

### Outcomes

*Students will be expected to*

F1 Continued...

#### Achievement Indicator:

F1.5 *Continued*

### Elaborations—Strategies for Learning and Teaching

Students should also analyze situations where regular loan payments are made over the term of the loan, such as, a mortgage or a car loan. Ask students why the frequency of a loan payment is often linked to a payroll schedule. Teachers should provide examples where the payment frequency matches the compounding period. For example, if a loan repayment is occurring monthly, then the interest should be compounded monthly as well. Problems where the compounding period and payment period differ will be explored later. Since the formula for a single loan payment does not apply here, either the payment amount must be given in the problem, or students should determine it using a financial application. Ask students to use technology to create an amortization table to illustrate each periodic payment on the loan. Students could work through a problem similar to the following:

- Mark is buying an ATV for the summer. The bank offers him a loan of \$7499.99 to pay for his ATV with an interest rate of 4.5% compounding monthly. If Mark makes 36 monthly payments of \$223.10, calculate the total interest paid at the end of the loan.

#/Year	Date	Payment	Interest	Principal	Balance
Loan:	12/01/12				7,499.99
1/01	01/01/13	223.10	28.12	194.98	7,305.01
2/01	02/01/13	223.10	27.39	195.71	7,109.30
3/01	03/01/13	223.10	26.66	196.44	6,912.86
4/01	04/01/13	223.10	25.92	197.18	6,715.68
5/01	05/01/13	223.10	25.18	197.92	6,517.76
6/01	06/01/13	223.10	24.44	198.66	6,319.10
7/01	07/01/13	223.10	23.70	199.40	6,119.70
8/01	08/01/13	223.10	22.95	200.15	5,919.55
9/01	09/01/13	223.10	22.20	200.90	5,718.65
10/01	10/01/13	223.10	21.44	201.66	5,516.99
11/01	11/01/13	223.10	20.69	202.41	5,314.58
12/01	12/01/13	223.10	19.93	203.17	5,111.41
13/02	01/01/14	223.10	19.17	203.93	4,907.48
14/02	02/01/14	223.10	18.40	204.70	4,702.78
15/02	03/01/14	223.10	17.64	205.46	4,497.32
16/02	04/01/14	223.10	16.86	206.24	4,291.08
17/02	05/01/14	223.10	16.09	207.01	4,084.07
18/02	06/01/14	223.10	15.32	207.78	3,876.29
19/02	07/01/14	223.10	14.54	208.56	3,667.73
20/02	08/01/14	223.10	13.75	209.35	3,458.38
21/02	09/01/14	223.10	12.97	210.13	3,248.25
22/02	10/01/14	223.10	12.18	210.92	3,037.33
23/02	11/01/14	223.10	11.39	211.71	2,825.62
24/02	12/01/14	223.10	10.60	212.50	2,613.12
25/03	01/01/15	223.10	9.80	213.30	2,399.82
26/03	02/01/15	223.10	9.00	214.10	2,185.72
27/03	03/01/15	223.10	8.20	214.90	1,970.82
28/03	04/01/15	223.10	7.39	215.71	1,755.11
29/03	05/01/15	223.10	6.58	216.52	1,538.59
30/03	06/01/15	223.10	5.77	217.33	1,321.26
31/03	07/01/15	223.10	4.95	218.15	1,103.11
32/03	08/01/15	223.10	4.14	218.96	884.15
33/03	09/01/15	223.10	3.32	219.78	664.37
34/03	10/01/15	223.10	2.49	220.61	443.76
35/03	11/01/15	223.10	1.66	221.44	222.32
36/03	12/01/15	223.15	0.83	222.32	0.00
	12/31/15	8,031.65	531.66	7,499.99	

Ask students to verify the values of the first two or three rows to ensure they understand how to calculate the interest, principal and present balance. They should then analyze the table to note the pattern in the interest calculation.

**General Outcome: Develop number sense in financial applications.****Suggested Assessment Strategies***Paper and Pencil*

- Brittany takes out a loan for \$100 000 at 6% annual interest. She takes 20 years to repay the loan. Ask students to answer to following:
  - (i) Use a financial application to determine the amount of each monthly payment.
  - (ii) How much interest will she have paid at the end of the 20 years?
- Tyler decides to purchase a BMX bike from the Cyclic Shop for \$2200 including taxes. He considers the following options:

Option A	Option B
-pay \$2200 in cash	- pay an initial administration fee of \$30 in cash - no downpayment - monthly payments of \$191.37 based on 8% annual interest compounded monthly for 1 year.

Ask students how much more will Tyler have to pay if he chooses Option B instead of Option A.

(F1.5)

*Performance*

- For *Pass the Problem*, groups of 5-6 students will complete a finance problem by creating an amortization table. Ask one student to complete the calculations for the first line of an amortization table. The second student will verify the calculations of the previous student, complete the next line and then pass the problem on. The amortization table keeps getting passed through the group until it is completed. The last person in the group is responsible for verifying all calculations by using a technology application.

(F1.5)

**Resources/Notes****Authorized Resource***Principles of Mathematics 12***10.1 Analyzing Loans**

SB: pp: 636-653

TR: pp.592-604

**Web Links**

This link provides instructions on how to use the finance application on the TI 83:

[www.distancemath.com/TI/Using%20the%20Finance%20Application%20on%20the%20TI%2083.pdf](http://www.distancemath.com/TI/Using%20the%20Finance%20Application%20on%20the%20TI%2083.pdf)

This link provides a mortgage amortization application in the form of a spreadsheet:

[www.office.microsoft.com/en-ca/templates/loan-amortization-schedule-TC001075591.aspx](http://www.office.microsoft.com/en-ca/templates/loan-amortization-schedule-TC001075591.aspx)

The following link provides an amortization calculator:

[www.gotcredit.com/amortizationloancalculator.php](http://www.gotcredit.com/amortizationloancalculator.php)

The following websites for financial institutions are listed:

[www.cibc.com](http://www.cibc.com)

[www.scotiabank.com](http://www.scotiabank.com)

[www.tdcanadatrust.com](http://www.tdcanadatrust.com)

[www.bmo.com](http://www.bmo.com)

## Financial Mathematics

### Outcomes

Students will be expected to

F1 Continued...

### Achievement Indicators:

F1.6 Determine, using technology, the total cost of a loan under a variety of conditions; e.g., different amortization periods, interest rates, compounding periods and terms.

F1.3 Continued

F1.7 Determine, using technology, the unknown variable in compound interest loan situations.

### Elaborations – Strategies for Learning and Teaching

Students should investigate the effects that changing a variable or multiple variables has on the total cost of a loan. They should first compare the effects of changing one specific variable. Using a financial application, ask students to complete the following table:

Principal	Interest Rate (Annual Compound)	Loan period (months)	Monthly payment	Total Cost of Loan	Total Interest (Total Cost-Principal)
\$18 000	4%	48			
\$18 000	4%	36			
\$18 000	3%	48			
\$18 000	3%	36			

Students should notice that by decreasing the amortization period, they are increasing their monthly payment but decreasing their total interest. Similar conclusions can be made regarding interest rates, compounding periods and terms.

Students should also investigate the effect that changing multiple conditions has on the total loan amount. For example, students can use a financial application to determine which of the following options for a mortgage results in a lower total loan cost.

- Option 1: Cost of mortgage is \$300 000, interest rate 2.5%, monthly payments, 25 year amortization.
- Option 2: Cost of mortgage is \$300 000, interest rate is 3.5%, monthly payments, 20 year amortization.

The variables in compound interest questions include: present value, regular payment amount, payment frequency, number of payments, annual interest rate, compounding frequency, total interest and future value. Questions should include:

- finding the present value, given all other variables.
- finding the number of payments, given all other variables.
- finding regular payment amount, given all other variables.
- finding future value, given all other variables

Students should be able to calculate the total interest at the end of the loan period.

**General Outcome: Develop number sense in financial applications.****Suggested Assessment Strategies***Journal*

- Ask students to visit the websites of at least three different financial institutions which offer mortgages (CIBC, Scotiabank, TD, BMO, etc.). They should decide which offers the best rate for a 5-year closed term mortgage with a fixed rate. Students should use the mortgage calculator on the website and determine for a 25-year amortization period

- the monthly payment amount
- total cost of the mortgage
- total amount of interest paid

Ask students to then change the payment frequency to accelerated biweekly. Ask them how this affects the three values determined previously? They should explain their reasoning.

(F1.6, F1.3)

*Performance*

- Arrange students into 4 or 5 groups and provide students with a financial problem similar to the following. Each group must first find the monthly payment and then will be assigned a different task to complete and will report their findings back to the class.

Sample Tasks:

Given a 25 year mortgage of \$250 000 at an annual interest rate of 3% compounded biweekly, ask students to complete the following:

- Find the total cost of the mortgage.
- Find the total interest paid at the end of the mortgage.
- Find the amount of principal paid in the first five years.
- Find the amount of principal paid in year one.
- Find the amount of principal paid in year 25.

(F1.6, F1.3)

**Resources/Notes****Authorized Resource***Principles of Mathematics 12***10.1 Analyzing Loans**

SB: pp: 636-653

TR: pp.592-604

**Web Links**

CMHC is an app that is useful for mortgages, car loans and personal loans.

Calc Apps (finance.calc) allows students to calculate information about all types of loans.

The following website allows students to determine the unknown variable given certain information about the loan.

[www.efunda.com](http://www.efunda.com)

This website helps create amortization tables to determine the total interest, the total amount paid, and the payment amount.

<http://www.pine-grove.com/online-calculators/amortization-schedule.htm>

The following website is a Canadian mortgage calculator.

[www.lendingmax.ca/calculator.php](http://www.lendingmax.ca/calculator.php)

## Financial Mathematics

### Outcomes

*Students will be expected to*

F1 Continued...

### Achievement Indicator:

F1.7 *Continued*

### Elaborations—Strategies for Learning and Teaching

Using technology, students have the power to make decisions about their ability to afford a loan based on their budget. Ask students to solve the following problems:

- Larry is considering buying a new home that costs \$220 000. He needs to know what the monthly payment would be for a 30 year mortgage at 4.75% interest compounded monthly.

Using a finance application on a graphing calculator or an online loan calculator, ask students to determine the payment amount (i.e., \$1147.62).

```
N=360
I%=4.75
PV=220000
PMT=-1147.62414
FV=0
P/Y=12
C/Y=12
PMT: [2ND] [PMT] BEGIN
```

Another example involves determining the maximum interest rate that a purchaser can afford for a car loan.

- Janet wants to buy a \$10 000 car and knows she can only afford a maximum monthly payment of \$200. She does not want to have the loan for any longer than five years. What is the maximum interest rate that she could afford to pay?

```
N=60
I%=0
PV=10000
PMT=-200
FV=0
P/Y=12
C/Y=12
PMT: [2ND] [PMT] BEGIN
```

Using a finance application on a graphing calculator or an online loan calculator, students conclude the highest rate Janet can afford is 7.4%. Ask students why this information is so important when making financial decisions. Teachers should discuss with students that this information, which is based on the budget of the individual, allows the person to negotiate a rate without exceeding the maximum.



**General Outcome: Develop number sense in financial applications.****Suggested Assessment Strategies***Performance*

- Pair of students play the game *Finance Tic-Tac-Toe*. Each group is given a deck of cards where each card contains a finance problem. A second deck of cards will provide the equation and solution. Cards could be labeled 1 – 9. Students lay the problem cards face down in a square 3 x 3 grid. They take turns flipping over a card and solving the problem using technology and checking the solution card. If someone solves a problem incorrectly, the other play will receive that square in the grid. However, before continuing on, the group should discuss the solution. The first player with three in a row wins.

(F1.7)

**Resources/Notes****Authorized Resource***Principles of Mathematics 12*

10.1 Analyzing Loans

SB: pp: 636-653

TR: pp.592-604

## Financial Mathematics

### Outcomes

Students will be expected to

F1 Continued...

#### Achievement Indicator:

F1.8 Compare and explain, using technology, different credit options that involve compound interest, including bank and store credit cards and special promotions.

### Elaborations – Strategies for Learning and Teaching

Students should work with forms of credit that include bank loans, credit cards and special promotions that have various conditions. They should compare credit options with varying interest rates, compounding periods, annual fees and special limited time offers such as "no interest" periods. Amortization tables, spreadsheets and financial applications should be used to determine monthly payments, total cost, total interest, etc., to ultimately determine the most financially sound investment.

Teachers should have a general discussion with students regarding the advantages and disadvantages of using line of credit, in-store financing options and credit cards for purchasing. Ask students to reflect on the following questions:

- Which borrowing method generally has the lowest interest rate?
- Are you planning on paying the balance off in full at the end each month or carrying forward a balance? How should this factor into your decision?
- Are there any promotions that you can take advantage of?
- Are there any 'hidden' fees?
- Should you pay more than the minimum required payment?
- Which borrowing option is best for large purchases?

To get a sense of how credit accounts work, students should begin by analyzing information in the two amortization tables in the following example: Brad and Marie decide to order a home gym online. The order totaled \$2668 and the shipping cost is \$347. They can afford to pay \$200 each month. Ask students which credit card should they use.

- Brad's credit card charges 13.9%, compounded daily, with an annual fee of \$75.
- Marie's credit card charges 19.8%, compounded daily.

(i) Repayment for Brad's Credit Card:

#/Year	Date	Payment	Interest	Principal	Balance
Loan:	12/01/12				3,015.00
1/01	01/01/13	200.00	37.20	162.80	2,927.20
2/01	02/01/13	200.00	35.24	164.76	2,762.44
3/01	03/01/13	200.00	30.02	169.98	2,592.46
4/01	04/01/13	200.00	31.21	168.79	2,423.67
5/01	05/01/13	200.00	28.23	171.77	2,251.90
6/01	06/01/13	200.00	27.11	172.89	2,079.01
7/01	07/01/13	200.00	24.22	175.78	1,903.23
8/01	08/01/13	200.00	22.91	177.09	1,726.14
9/01	09/01/13	200.00	20.78	179.22	1,546.92
10/01	10/01/13	200.00	18.02	181.98	1,364.94
11/01	11/01/13	200.00	16.43	183.57	1,181.37
12/01	12/01/13	200.00	13.76	186.24	995.13
Y-T-D 2013	12/31/13	2,400.00	305.12	2,094.87	
Running	12/31/13	2,400.00	305.13	2,094.87	
13/02	01/01/14	200.00	11.98	188.02	807.11
14/02	02/01/14	200.00	9.72	190.28	616.83
15/02	03/01/14	200.00	6.70	193.30	423.53
16/02	04/01/14	200.00	5.10	194.90	228.63
17/02	05/01/14	231.29	2.66	228.63	0.00
Y-T-D 2014	12/31/14	1,031.29	36.16	995.13	
Running	12/31/14	3,431.29	341.29	3,090.00	

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**General Outcome: Develop number sense in financial applications.**


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**Suggested Assessment Strategies**
*Journal*

- Lori is thinking about getting a credit card. She receives a phone call in which a credit card sales representative tells her she has been pre-approved for a credit card. Ask students to list five questions that Lori should ask about before she accepts the offer.
- Ask students to create a graphic organizer that lists the pros and cons for 4 different types of credit, such as, bank credit cards, store credit cards, line of credit, in-store financing, payday credit, personal loans.

(F1.8)

*Performance*

- The online game *Financial Football* can be played individually or in pairs and uses financial questions to allow team movement down the field to score touchdowns. The more questions students answer correctly the more likely they are to win.

(F1.3, F1.6, F1.7, F1.8)

**Resources/Notes**
**Authorized Resource**
*Principles of Mathematics 12*
**10.2 Exploring Credit Card Use**

SB: pp: 654-656

TR: pp.605-610

**10.3 Solve Problems Involving Credit**

SB: pp: 660-675

TR: pp.612-619

**Web Link**

The link for Financial Football:

*<http://www.practicalmoneyskills.com/games/trainingcamp/fff>*

**Mark Money** is an app that allows students to calculate the different variables in a compound interest loan situation.

**Financial Mathematics**

**Outcomes**

*Students will be expected to*

F1 Continued...

**Achievement Indicator:**

F1.8 *Continued*

**Elaborations – Strategies for Learning and Teaching**

(ii) Repayment for Marie's Credit Card:

#/Year	Date	Payment	Interest	Principal	Balance
Loan:	12/01/12				3,015.00
1/01	01/01/13	200.00	51.83	148.17	2,866.83
2/01	02/01/13	200.00	49.28	150.72	2,716.11
3/01	03/01/13	200.00	42.14	157.86	2,558.25
4/01	04/01/13	200.00	43.98	156.02	2,402.23
5/01	05/01/13	200.00	39.95	160.05	2,242.18
6/01	06/01/13	200.00	38.55	161.45	2,080.73
7/01	07/01/13	200.00	34.61	165.39	1,915.34
8/01	08/01/13	200.00	32.93	167.07	1,748.27
9/01	09/01/13	200.00	30.06	169.94	1,578.33
10/01	10/01/13	200.00	26.25	173.75	1,404.58
11/01	11/01/13	200.00	24.15	175.85	1,228.73
12/01	12/01/13	200.00	20.44	179.56	1,049.17
-D 2013	12/31/13	2,400.00	434.17	1,965.82	
Running	12/31/13	2,400.00	434.17	1,965.83	
13/02	01/01/14	200.00	18.04	181.96	867.21
14/02	02/01/14	200.00	14.91	185.09	682.12
15/02	03/01/14	200.00	10.58	189.42	492.70
16/02	04/01/14	200.00	8.47	191.53	301.17
17/02	05/01/14	306.18	5.01	301.17	0.00

Teachers could use the following questions to promote student discussion:

- (i) After the third payment, which credit card option appears to be better? Do you think this will always be the case?
- (ii) At the end of the sixth month, which credit card appears to be better? Discuss your findings.
- (iii) Overall, which credit card was the better choice and why?

Ask students to comment on the impact of the annual fee on the determination of which card is better.

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**General Outcome: Develop number sense in financial applications.**


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**Suggested Assessment Strategies**
*Guest Speaker*

- Invite a guest speaker such as an accountant, banker, mortgage specialist or financial advisor, to generate classroom discussion around borrowing and investing.

(F1.8)

**Resources/Notes**
**Authorized Resource**
*Principles of Mathematics 12*
**10.2 Exploring Credit Card Use**

SB: pp: 654-656

TR: pp.605-610

**10.3 Solve Problems Involving Credit**

SB: pp: 660-675

TR: pp.612-619

**Web Link**

**Easy Loan Calculator App** allows students to calculate all types of loans with all types of interest.

## Financial Mathematics

### Outcomes

*Students will be expected to*

**F2 Analyze costs and benefits of renting, leasing and buying.**

[CN, PS, R, T]

#### Achievement Indicators:

F2.1 *Identify and describe examples of assets that appreciate or depreciate.*

F2.2 *Compare, using examples, renting, leasing and buying.*

F2.3 *Justify, for a specific set of circumstances, if renting, buying or leasing would be advantageous.*

F2.4 *Solve, using technology, a contextual problem that involves cost-and-benefit analysis.*

### Elaborations—Strategies for Learning and Teaching

Students should solve problems that involve decisions about whether to buy, rent, or lease in a variety of contexts.

Appreciation and depreciation both deal with asset value over time. Assets such as real estate and bonds usually gain value over time while other assets such as vehicles may decrease in value. Ask students to think about situations where assets can appreciate or depreciate, and what factors contribute to this. Houses usually appreciate over time, for example, but there are economic circumstances that may cause them to depreciate. Likewise, vehicles can appreciate or depreciate. Students could discuss why they should factor this in when they are deciding to buy a new car versus used car.

When deciding to rent, lease or buy, completing a cost-and-benefit analysis is essential in determining which option is best. To make an informed decision, ask students to consider:

- affordable monthly payments
- amount of disposable income
- the importance of building equity
- interest rates
- initial fees and possible penalties
- amount of downpayment
- total end cost including interest
- personal benefits such as convenience and flexibility
- appreciation and depreciation
- amount of - amount of disposable income (ie money left when bills are all paid)
- the importance of building equity (ie the portion of the house that you own once the mortgage is paid off)
- initial fees and possible penalties (ie repaying a mortgage early, lawyers fees, property tax, etc.,...)

The following example is a good starting point to present to students:

- Sarah is going to university in the fall and her parents are trying to decide whether to buy a house or rent an apartment for the 5 years she will be there. The house they are considering buying costs \$200,000 and requires a down payment of \$10,000. The bank will provide a 20-year mortgage, for the remainder of the cost, at 3% compounded semi-annually with payments every month. The house they are considering renting is \$1400 per month and requires an initial damage deposit of \$700.

**General Outcome: Develop number sense in financial applications.****Suggested Assessment Strategies***Paper and Pencil*

- Ask students to complete the following Frayer Model for assets that appreciate and/or depreciate.

**Frayer Model**

Definition	Characteristics
Examples	Non-examples

WORD

(F2.1)

- Colleen participates in a field trip where she skies for the first time. She really enjoys the experience and decides that she will enrol in ski lessons everyday for two weeks and then hopefully continue skiing for years to come. She needs to decide if she is going to purchase her own skies or rent them. It costs \$25 a day to rent skies while it costs \$525 to buy them.
  - (i) For two weeks how much would it cost to rent skis?
  - (ii) How much would it cost to buy them?
  - (iii) For the two week period, is it more economical to rent or buy the skis?
  - (iv) After how many days will the rental cost equal the cost of buying?
  - (v) Would you recommend Colleen buy or rent the skis? What are some of the things she should consider in making her decision?

(F2.3)

**Resources/Notes****Authorized Resource***Principles of Mathematics 12***10.4 Buy, Rent, or Lease?**

SB: pp: 676-689

TR: pp.620-626

## Financial Mathematics

### Outcomes

*Students will be expected to*

F2 Continued

### Achievement Indicators:

F2.2 & F2.3 *Continued*

### Elaborations – Strategies for Learning and Teaching

Ask students to answer the following questions:

- (i) What is the monthly payment for the mortgage?
- (ii) What is the total amount spent in the first five years if you purchase the house?
- (iii) What is the monthly payment for renting?
- (iv) What is the total cost for renting for 5 years?
- (v) In the first five years, how much has been paid on the principal of the mortgage?
- (vi) If we only consider monthly costs, which option would be best?
- (vii) What other factors (ex: equity, maintenance, initial fees, penalties) should a person consider when making this decision?

Teachers should discuss the factors involved when deciding to rent, lease or buy a home. The difference between renting and leasing a home is the length of time a property is being rented.

Buy	Rent	Lease
<ul style="list-style-type: none"> <li>• build equity</li> <li>• long term investment</li> <li>• renovate the house</li> <li>• maintenance costs</li> </ul>	<ul style="list-style-type: none"> <li>• can be short term</li> <li>• can be contract free</li> </ul>	<ul style="list-style-type: none"> <li>• long term investment</li> <li>• includes a contract</li> <li>• must pay in full if the person decides to move before the end of the contract</li> </ul>
	<ul style="list-style-type: none"> <li>• do not own</li> <li>• cannot change the property</li> <li>• rent can change after 12 months</li> <li>• owner's rules</li> <li>• no maintenance costs</li> </ul>	

Discuss the factors involved when deciding to lease or buy a car:

Buy	Lease
<ul style="list-style-type: none"> <li>• build equity</li> <li>• long term investment</li> <li>• warranty</li> <li>• unlimited mileage</li> </ul>	<ul style="list-style-type: none"> <li>• do not own</li> <li>• maintenance costs</li> <li>• generally lower payments</li> <li>• warranty</li> <li>• option to buy at the end of the lease</li> <li>• inspection upon return</li> <li>• limited mileage</li> </ul>



**General Outcome: Develop number sense in financial applications.****Suggested Assessment Strategies***Performance*

- Marie is unsure whether she should buy or lease a new vehicle. Ask students to choose two comparable vehicles, each from a different manufacturer (i.e., Toyota Corolla and Honda Civic). Ask them to visit the manufacturers' websites or use advertisements to obtain information on the price. They should use online buying calculators to answer the following questions:
  - (i) Which has the lower purchase price?
  - (ii) If buying over a 60 month period, which is the better deal?
  - (iii) Which is better for a 60 month lease?
  - (iv) Which car and payment option do you think is the better deal? Why? (Note: Be sure to consider vehicle features, interest rates, value after 5 years, etc.,)

(F2.2, F2.4)

*Journal*

- Ask students to respond to the following:
  - (i) Create a foldable or graphic organizer to compare leasing versus buying a car.
 

(F2.2, F2.3)
  - (ii) Describe a situation where renting a house is the best option.
 

(F2.3)

**Resources/Notes****Authorized Resource***Principles of Mathematics 12***10.4 Buy, Rent, or Lease?**

SB: pp: 676-689

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**Web Links**

The following site provides information regarding residential tenancies:

[http://www.servicenl.gov.nl.ca/landlord/residential\\_tenancies.html](http://www.servicenl.gov.nl.ca/landlord/residential_tenancies.html)

[http://www.servicenl.gov.nl.ca/landlord/rental\\_agreement.pdf](http://www.servicenl.gov.nl.ca/landlord/rental_agreement.pdf)



## *Appendix:*

Outcomes with Achievement Indicators  
Organized by Topic  
(With Curriculum Guide References)

Topic: Logical Reasoning	General Outcome: Develop logical reasoning.	
Specific Outcomes	Achievement Indicators	Page Reference
It is expected that students will:	<i>The following sets of indicators help determine whether students have met the corresponding specific outcome</i>	
<p>LR1. Analyze puzzles and games that involve numerical and logical reasoning, using problem-solving strategies.</p> <p>[C, CN, ME, PS, R]</p>	<p>LR1.1 Determine, explain and verify a strategy to solve a puzzle or to win a game; e.g.,</p> <ul style="list-style-type: none"> <li>• guess and check</li> <li>• look for a pattern</li> <li>• make a systematic list</li> <li>• draw or model</li> <li>• eliminate possibilities</li> <li>• simplify the original problem</li> <li>• work backward</li> <li>• develop alternative approaches.</li> </ul> <p>LR1.2 Identify and correct errors in a solution to a puzzle or in a strategy for winning a game.</p> <p>LR1.3 Create a variation on a puzzle or a game, and describe a strategy for solving the puzzle or winning the game.</p>	<p>p. 38-47. p. 74, p. 114, p. 184</p>
<p>LR2. Solve problems that involve the application of set theory.</p> <p>[CN, PS, R, V]</p>	<p>LR2.1 Provide examples of the empty set, disjoint sets, subsets and universal sets in context, and explain the reasoning.</p> <p>LR2.2 Organize information such as collected data and number properties, using graphic organizers, and explain the reasoning.</p> <p>LR2.3 Explain what a specified region in a Venn diagram represents, using connecting words (and, or, not) or set notation.</p> <p>LR2.4 Determine the elements in the complement, the intersection or the union of two sets.</p> <p>LR2.5 Explain how set theory is used in applications such as Internet searches, database queries, data analysis, games and puzzles.</p> <p>LR2.6 Identify and correct errors in a solution to a problem that involves sets.</p> <p>LR2.7 Solve a contextual problem that involves sets, and record the solution, using set notation.</p>	<p>p. 22-24</p> <p>p. 24-28</p> <p>p. 26 - 28</p> <p>p. 26-28</p> <p>p. 30-32 p. 36</p> <p>p. 34</p> <p>p. 34-36</p>

Topic: Probability	General Outcome: Develop critical thinking skills related to uncertainty.	
Specific Outcomes	Achievement Indicators	Page Reference
It is expected that students will:	<i>The following sets of indicators help determine whether students have met the corresponding specific outcome</i>	
P1. Interpret and assess the validity of odds and probability statements. [C, CN, ME]	P1.1 Provide examples of statements of probability and odds found in fields such as media, biology, sports, medicine, sociology and psychology. P1.2 Explain, using examples, the relationship between odds (part-part) and probability (part-whole). P1.3 Express odds as a probability and vice versa. P1.4 Determine the probability of, or the odds for and against, an outcome in a situation. P1.5 Explain, using examples, how decisions may be based on probability or odds and on subjective judgments. P1.6 Solve a contextual problem that involves odds or probability.	p. 80  p. 80  p. 80  p. 82  p. 82  p. 82
P2. Solve problems that involve the probability of mutually exclusive and non-mutually exclusive events. [CN, PS, R, V]	P2.1 Classify events as mutually exclusive or non-mutually exclusive, and explain the reasoning. P2.2 Determine if two events are complementary, and explain the reasoning. P2.3 Represent, using set notation or graphic organizers, mutually exclusive (including complementary) and non-mutually exclusive events. P2.4 Solve a contextual problem that involves the probability of mutually exclusive or non-mutually exclusive events. P2.5 Create and solve a problem that involves mutually exclusive or non-mutually exclusive events.	p. 88  p. 88  p. 88  p. 88-90  p. 88-90
P3. Solve problems that involve the probability of two events. [CN, PS, R]	P3.1 Compare, using examples, dependent and independent events. P3.2 Determine the probability of two independent events. P3.3 Determine the probability of an event, given the occurrence of a previous event. P3.4 Create and solve a contextual problem that involves determining the probability of dependent or independent events.	p. 92  p. 92  p. 94  p. 94

Topic: Probability	General Outcome: Develop critical thinking skills related to uncertainty.	
Specific Outcomes	Achievement Indicators	Page Reference
It is expected that students will:	<i>The following sets of indicators help determine whether students have met the corresponding specific outcome</i>	
P4. Solve problems that involve the Fundamental Counting Principle. [PS, R, V]	P4.1 Represent and solve counting problems, using a graphic organizer. P4.2 Generalize, using inductive reasoning, the Fundamental Counting Principle. P4.3 Identify and explain assumptions made in solving a counting problem. P4.4 Solve a contextual counting problem, using the Fundamental Counting Principle, and explain the reasoning.	p. 52-56  p. 52  p. 52-56  p. 52-56
P5. Solve problems that involve permutations. [ME, PS, R, T, V]	P5.1 Represent the number of arrangements of $n$ elements taken $n$ at a time, using factorial notation. P5.2 Determine, with or without technology, the value of a factorial. P5.3 Simplify a numeric or an algebraic fraction that contains factorials in both the numerator and denominator. P5.4 Solve an equation that involves factorials. P5.5 Determine the number of permutations of $n$ elements taken $r$ at a time. P5.6 Generalize strategies for determining the number of permutations of $n$ elements taken $r$ at a time. P5.7 Determine the number of permutations of $n$ elements taken $n$ at a time where some elements are not distinct. P5.8 Explain, using examples, the effect on the total number of permutations of $n$ elements when two or more elements are identical. P5.9 Solve a contextual problem that involves probability and permutations.	p. 58-60  p. 58-60  p. 60  p. 60, p. 66, p. 74  p. 62-66  p. 62-66  p. 68  p. 68  p. 84-86

<b>Topic:</b> Probability	<b>General Outcome:</b> Develop critical thinking skills related to uncertainty.	
<b>Specific Outcomes</b>	<b>Achievement Indicators</b>	<b>Page Reference</b>
It is expected that students will:	<i>The following sets of indicators help determine whether students have met the corresponding specific outcome</i>	
P6. Solve problems that involve combinations. [ME, PS, R, T, V]	P6.1 Explain, using examples, why order is or is not important when solving problems that involve permutations or combinations.	p. 70
	P6.2 Determine the number of combinations of $n$ elements taken $r$ at a time.	p. 70-72
	P6.3 Generalize strategies for determining the number of combinations of $n$ elements taken $r$ at a time.	p. 70-72
	P6.4 Solve a contextual problem that involves probability and combinations.	p. 74, p. 84-86

<b>Topic:</b> Relations and Functions	<b>General Outcome:</b> Develop algebraic and graphical reasoning through the study of relations.	
<b>Specific Outcomes</b>	<b>Achievement Indicators</b>	<b>Page Reference</b>
It is expected that students will:	<i>The following sets of indicators help determine whether students have met the corresponding specific outcome</i>	
RF1. Determine equivalent forms of rational expressions (limited to numerators and denominators that are monomials and binomials). [C, ME, R]	RF1.1 Explain why a given value is non-permissible for a given rational expression.	p. 100-102
	RF1.2 Determine the non-permissible values for a rational expression.	p. 100-102
	RF1.3 Compare the strategies for writing equivalent forms of rational expressions to writing equivalent forms of rational numbers.	p. 102
	RF1.4 Determine a rational expression that is equivalent to a given rational expression by multiplying the numerator and denominator by the same factor (limited to a monomial or a binomial), and state the non-permissible values of the equivalent rational expression.	p. 102
	RF1.5 Simplify a rational expression.	p. 104
	RF1.6 Explain why the non-permissible values of a given rational expression and its simplified form are the same.	p. 104
	RF1.7 Identify and correct errors in a given simplification of a rational expression, and explain the reasoning.	p. 104

<b>Topic:</b> Relations and Functions	<b>General Outcome:</b> Develop algebraic and graphical reasoning through the study of relations.	
<b>Specific Outcomes</b>	<b>Achievement Indicators</b>	<b>Page Reference</b>
It is expected that students will:	<i>The following sets of indicators help determine whether students have met the corresponding specific outcome</i>	
RF2. Perform operations on rational expressions (limited to numerators and denominators that are monomials and binomials).	RF2.1 Compare the strategies for performing a given operation on rational expressions to the strategies for performing the same operation on rational numbers.	p. 106
[CN, ME, R]	RF2.2 Determine the non-permissible values when performing operations on rational expressions.	p. 106-108
	RF2.3 Determine, in simplified form, the product or quotient of two rational expressions.	p. 106
	RF2.4 Determine, in simplified form, the sum or difference of rational expressions that have the same denominator.	p. 108
	RF2.5 Determine, in simplified form, the sum or difference of two rational expressions that have different denominators.	p. 108
RF3. Solve problems that involve rational equations (limited to numerators and denominators that are monomials and binomials).	RF3.1 Determine the non-permissible values for the variable in a rational equation.	p. 110-112
[C, CN, PS, R]	RF3.2 Determine the solution to a rational equation algebraically, and explain the strategy used to solve the equation.	p. 110-112
	RF3.3 Explain why a value obtained in solving a rational equation may not be a solution of the equation.	p. 110-112
	RF3.4 Solve a contextual problem that involves a rational equation.	p. 112-114



<b>Topic:</b> Relations and Functions	<b>General Outcome:</b> Develop algebraic and graphical reasoning through the study of relations.	
<b>Specific Outcomes</b> It is expected that students will:	<b>Achievement Indicators</b> <i>The following sets of indicators help determine whether students have met the corresponding specific outcome</i>	<b>Page Reference</b>
RF4. Demonstrate an understanding of logarithms and the laws of logarithms. [C, CN, ME, R]	<p>RF4.1 Express a logarithmic equation as an exponential equation and vice versa.</p> <p>RF4.2 Determine the value of a logarithmic expression, such as <math>\log_2 8</math>, without technology.</p> <p>RF4.3 Develop the laws of logarithms, using numeric examples and the exponent laws.</p> <p>RF4.4 Determine an equivalent expression for a logarithmic expression by applying the laws of logarithms.</p> <p>RF4.5 Determine the approximate value of a logarithmic expression, such as <math>\log_2 9</math>, with technology.</p>	<p>p. 170</p> <p>p. 170</p> <p>p. 172-174</p> <p>p. 174</p> <p>p. 180</p>
RF5. Solve problems that involve exponential equations. [C, CN, PS, R, T]	<p>RF5.1 Determine the solution of an exponential equation in which the bases are powers of one another; e.g., <math>2^{x-1} = 4^{x-2}</math>.</p> <p>RF5.2 Determine the solution of an exponential equation in which the bases are not powers of one another; e.g., <math>2^{x-1} = 3^{x+1}</math>.</p> <p>RF5.3 Solve problems that involve the application of exponential equations.</p> <p>RF5.4 Solve problems that involve logarithmic scales, such as the Richter scale and the pH scale.</p>	<p>p. 142-146, p.150</p> <p>p. 146-148, p. 178</p> <p>p. 150, p.156-158, p. 182</p> <p>p. 176</p>

<b>Topic:</b> Relations and Functions	<b>General Outcome:</b> Develop algebraic and graphical reasoning through the study of relations.	
<b>Specific Outcomes</b> It is expected that students will:	<b>Achievement Indicators</b> <i>The following sets of indicators help determine whether students have met the corresponding specific outcome</i>	<b>Page Reference</b>
RF6. Represent data, using exponential and logarithmic functions, to solve problems. [C, CN, PS, T, V]	RF6.1 Describe, orally and in written form, the characteristics of an exponential function by analyzing its graph.	p. 138
	RF6.2 Describe, orally and in written form, the characteristics of an exponential function by analyzing its equation.	p. 140
	RF6.3 Match exponential equations in a given set to their corresponding graphs.	p. 142
	RF6.4 Graph data, and determine the exponential function that best approximates the data.	p. 152-158
	RF6.5 Interpret the graph of an exponential function that models a situation, and explain the reasoning.	p. 152-158
	RF6.6 Solve, using technology, a contextual problem that involves data that is best represented by graphs of exponential functions and explain the reasoning.	p. 152-158
	RF6.7 Describe, orally and in written form, the characteristics of a logarithmic function by analyzing its graph.	p. 164-168
	RF6.8 Describe, orally and in written form, the characteristics of a logarithmic function by analyzing its equation.	p. 164-168
	RF6.9 Match equations in a given set to their corresponding graphs.	p. 168
	RF6.10 Graph data, and determine the logarithmic function that best approximates the data.	p. 184
	RF6.11 Interpret the graph of a logarithmic function that models a situation, and explain the reasoning.	p. 184
	RF6.12 Solve, using technology, a contextual problem that involves data that is best represented by graphs of logarithmic functions and explain the reasoning.	p. 184

<b>Topic:</b> Relations and Functions	<b>General Outcome:</b> Develop algebraic and graphical reasoning through the study of relations.	
<b>Specific Outcomes</b>	<b>Achievement Indicators</b>	<b>Page Reference</b>
It is expected that students will:	<i>The following sets of indicators help determine whether students have met the corresponding specific outcome</i>	
RF7. Represent data, using polynomial functions (of degree $\leq 3$ ), to solve problems. [C, CN, PS, T, V]	RF7.1 Describe, orally and in written form, the characteristics of a polynomial function by analyzing its graph. RF7.2 Describe, orally and in written form, the characteristics of a polynomial function by analyzing its equation. RF7.3 Match equations in a given set to their corresponding graphs. RF7.4 Graph data, and determine the polynomial function that best approximates the data. RF7.5 Interpret the graph of a polynomial function that models a situation, and explain the reasoning. RF7.6 Solve, using technology, a contextual problem that involves data that is best represented by graphs of polynomial functions, and explain the reasoning.	p. 120-124  p. 126  p. 128  p. 130-132  p.130-132  p.130-132
RF8. Represent data, using sinusoidal functions, to solve problems. [C, CN, PS, T, V]	RF8.1 Demonstrate an understanding of angles expressed in degrees and radians. RF8.2 Describe, orally and in written form, the characteristics of a sinusoidal function by analyzing its graph. RF8.3 Interpret the graph of a sinusoidal function that models a situation, and explain the reasoning. RF8.4 Describe, orally and in written form, the characteristics of a sinusoidal function by analyzing its equation. RF8.5 Match equations in a given set to their corresponding graphs. RF8.6 Graph data, and determine the sinusoidal function that best approximates the data. RF8.7 Solve, using technology, a contextual problem that involves data that is best represented by graphs of sinusoidal functions, and explain the reasoning	p. 190-192  p. 194  p. 196  p. 196-200  p. 200  p. 202  p. 202

Topic: Financial Mathematics	General Outcome: Develop number sense in financial applications.	
Specific Outcomes	Achievement Indicators	Page Reference
It is expected that students will:	<i>The following sets of indicators help determine whether students have met the corresponding specific outcome</i>	
F1 Solve problems that involve compound interest in financial decision making. [C, CN, PS, T, V]	F1.1 Explain the advantages and disadvantages of compound interest and simple interest. F1.2 Identify situations that involve compound interest. F1.3 Solve a contextual problem that involves compound interest. F1.4 Compare, in a given situation, the total interest paid or earned for different compounding periods. F1.5 Determine the total interest of a loan given the principal, interest rate and number of compounding periods. F1.6 Determine, using technology, the total cost of a loan under a variety of conditions; e.g., different amortization periods, interest rates, compounding periods and terms. F1.7 Determine, using technology, the unknown variable in compound interest loan situations. F1.8 Compare and explain, using technology, different credit options that involve compound interest, including bank and store credit cards and special promotions.	p. 208-210 p. 208-210 p. 208-210 p. 210-212 p.210-214 p. 216 p. 216-218 p. 220-222
F2 Analyze costs and benefits of renting, leasing and buying. [CN, PS, R, T]	F2.1 Identify and describe examples of assets that appreciate or depreciate. F2.2 Compare, using examples, renting, leasing and buying. F2.3 Justify, for a specific set of circumstances, if renting, buying or leasing would be advantageous. F2.4 Solve, using technology, a contextual problem that involves cost-and-benefit analysis.	p. 224 p. 224-226 p. 224-226 p. 224

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